Problem Seminar, Fall 2020
Mean Value Theorems

1. Let $f : [0,1] \to \mathbb{R}$ be a continuous function satisfying $\int_0^1 f(x)\,\mathrm{d}x = 0$. Prove that there is some $c \in (0,1)$ such that
$$(1-c)f(c) = c\int_0^c f(x)\,\mathrm{d}x.$$

2. Let $f : [0,1] \to \mathbb{R}$ be twice differentiable. Suppose that the line segment joining the points $(0, f(0))$ and $(1, f(1))$ intersects the graph of f at $(a, f(a))$ for some $a \in (0,1)$. Show that there exists $c \in (0,1)$ such that $f''(c) = 0$.

3. Let $f : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b). Show that there exist $c_1, c_2 \in (a,b)$ such that
$$\frac{f'(c_1)}{a+b} = \frac{f'(c_2)}{2c_2}.$$

4. Show that
$$1 - \frac{x^2}{2} < \cos x \text{ for } x \neq 0.$$
Hint: Use Cauchy's MVT.

5. Let $f : \mathbb{R} \to \mathbb{R}$ twice differentiable with $f''(x) > 0$ for all $x \in \mathbb{R}$. Show that
$$f(x + f'(x)) \geq f(x) \text{ for all } x \in \mathbb{R}.$$
Hint: Take three cases, $f'(x) = 0, f'(x) < 0$ and $f'(x) > 0$. For the last two cases use MVT.