INEQUALITIES

1. Classical inequalities

(1) The Cauchy-Schwarz inequality:

\[
\left( \sum_{k=1}^{n} a_k b_k \right)^2 \leq \left( \sum_{k=1}^{n} a_k^2 \right) \left( \sum_{k=1}^{n} b_k^2 \right)
\]

and integral version

\[
\left( \int_{a}^{b} f(x) g(x) \, dx \right)^2 \leq \left( \int_{a}^{b} f^2(x) \, dx \right) \left( \int_{a}^{b} g^2(x) \, dx \right).
\]

(2) Hölder inequality:

\[
\left| \sum_{k=1}^{n} a_k b_k \right| \leq \left( \sum_{k=1}^{n} |a_k|^p \right)^{\frac{1}{p}} \left( \sum_{k=1}^{n} |b_k|^q \right)^{\frac{1}{q}}
\]

and integral version

\[
\int_{a}^{b} |f(x) g(x)| \, dx \leq \left( \int_{a}^{b} |f|^p(x) \, dx \right)^{\frac{1}{p}} \left( \int_{a}^{b} |g|^q(x) \, dx \right)^{\frac{1}{q}},
\]

where \( \frac{1}{p} + \frac{1}{q} = 1 \) and \( p, q > 1 \).

(3) Minkowski inequality

\[
\left( \sum_{k=1}^{n} |a_k + b_k|^p \right)^{\frac{1}{p}} \leq \left( \sum_{k=1}^{n} |a_k|^p \right)^{\frac{1}{p}} + \left( \sum_{k=1}^{n} |b_k|^p \right)^{\frac{1}{p}}
\]

and integral version

\[
\int_{a}^{b} |f(x) + g(x)|^p \, dx \leq \left( \int_{a}^{b} |f|^p(x) \, dx \right)^{\frac{1}{p}} + \left( \int_{a}^{b} |g|^p(x) \, dx \right)^{\frac{1}{p}}.
\]
(4) Jensen inequality:

$$f \left( \frac{x_1 + \ldots + x_n}{n} \right) \leq \frac{f(x_1) + \ldots + f(x_n)}{n}$$

if $f$ is convex. Integral version

$$f \left( \int_0^1 u(x) \, dx \right) \leq \int_0^1 f(u(x)) \, dx,$$

for any continuous function $u$, if $f$ is convex.

(5) Young’s inequality:

$$ab \leq \int_0^a f(x) \, dx + \int_0^b f^{-1}(x) \, dx,$$

if $f : [0, \infty) \to [0, \infty)$ is continuous, strictly increasing and $f(0) = 0$.

(6) Chebyshev inequality

$$\left( \sum_{k=1}^n a_k \right) \left( \sum_{k=1}^n b_k \right) \leq n \sum_{k=1}^n a_k b_k$$

if $(a_k)_k$ and $(b_k)_k$ are increasing sequences. Integral version

$$\left( \int_a^b f(x) \, dx \right) \left( \int_a^b g(x) \, dx \right) \leq (b-a) \int_a^b f(x) g(x) \, dx$$

if $f$ and $g$ are both increasing functions.

1.1. Examples using classical inequalities. Solve the following.

1. The AM-GM inequality

$$\sqrt[n]{a_1 \ldots a_n} \leq \frac{a_1 + \ldots + a_n}{n}$$

for all positive numbers $a_1, \ldots, a_n$.

Proof. Use Jensen for $f(x) = \ln x$, which is concave on $(0, \infty)$. □
2. Prove that
\[ ab \leq \frac{a^p}{p} + \frac{b^q}{q} \]
for any \( a, b > 0 \), where \( \frac{1}{p} + \frac{1}{q} = 1 \) and \( p, q > 1 \).

Proof. Apply Young’s inequality to \( f(x) = x^{p-1} \), for \( p > 1 \). \( \square \)

3. Let \( x_i \in (0, \pi) \) and \( x = \frac{x_1 + \ldots + x_n}{n} \). Prove that
\[ \prod_{i=1}^{n} \frac{\sin x_i}{x_i} \leq \left( \frac{\sin x}{x} \right)^n. \]

Proof. Taking log, we see that the statement to prove is equivalent to
\[ \frac{1}{n} \sum_{i=1}^{n} \frac{\sin x_i}{x_i} \leq \ln \left( \frac{\sin x}{x} \right). \]
The function \( f(t) = \ln \left( \frac{\sin t}{t} \right) \) is concave on \( (0, \pi) \), so by Jensen we have
\[ \frac{f(x_1) + \ldots + f(x_n)}{n} \leq f \left( \frac{x_1 + \ldots + x_n}{n} \right). \]
This implies (1). \( \square \)

4. Prove that
\[ (a_1 \ldots a_n)^{\frac{1}{n}} + (b_1 \ldots b_n)^{\frac{1}{n}} \leq (a_1 + b_1 \ldots (a_n + b_n)) \]
for any positive numbers \( a_1, \ldots, a_n \) and \( b_1, \ldots, b_n \).

Proof. Scaling trick: the inequality to be proved does not change if we replace \( a_i \) by \( \lambda_i a_i \) and \( b_i \) by \( \lambda_i b_i \), where \( \lambda_i > 0 \).

So we may assume that
\[ a_i + b_i = 1, \]
for all \( i = 1, 2, \ldots, n \), by simultaneously rescaling \( a_i \) and \( b_i \) by an appropriate \( \lambda_i \).

Then the inequality to be proved becomes
\[ (a_1 \ldots a_n)^{\frac{1}{n}} + (b_1 \ldots b_n)^{\frac{1}{n}} \leq 1. \]
Using the AM-GM inequality we have
\[ (a_1 \ldots a_n)^{\frac{1}{n}} \leq \frac{a_1 + \ldots + a_n}{n} \]
\[ (b_1 \ldots b_n)^{\frac{1}{n}} \leq \frac{b_1 + \ldots + b_n}{n} \]
so that adding up we get
\[ (a_1 \ldots a_n)^{\frac{1}{n}} + (b_1 \ldots b_n)^{\frac{1}{n}} \leq \frac{(a_1 + b_1) + \ldots + (a_n + b_n)}{n} \]
\[ = 1 \]
where the second line follows from (2).

5. Prove that
\[ x^y y^z z^x \geq (xyz)^{\frac{x+y+z}{3}} \]
for any positive numbers \( x, y, z \).

**Proof.** Without loss of generality, we may assume that \( x \leq y \leq z \). Taking log, we see that the inequality to be proved is equivalent to
\[ (x + y + z) \ln x + \ln y + \ln z \leq 3 (x \ln x + y \ln y + z \ln z). \]
Since \( x \leq y \leq z \) and \( \ln x \leq \ln y \leq \ln z \), the above inequality follows from Chebyshev. \( \square \)

2. SOME TECHNIQUES OF PROVING INEQUALITIES

Not all new inequalities reduce to the classical ones. Sometimes we need new ideas.

2.1. Rearrangement inequality.

**Lemma 1.** If \( a_1 \leq a_2 \leq a_3 \) and \( b_1 \leq b_2 \leq b_3 \) then
\[ a_1 x_1 + a_2 x_2 + a_3 x_3 \leq a_1 b_1 + a_2 b_2 + a_3 b_3 \]
for any permutation \((x_1, x_2, x_3)\) of \((b_1, b_2, b_3)\).

The result is true for increasing sequences of any number of terms.

1. Prove that
\[ \frac{a}{b+c} + \frac{b}{a+c} + \frac{c}{a+b} \geq \frac{3}{2} \]
for any \( a, b, c > 0 \).

**Proof.** Without loss of generality we may assume \( a \leq b \leq c \). Then \((a, b, c)\) and \((\frac{1}{a+c}, \frac{1}{a+c}, \frac{1}{a+c})\) are similarly arranged, so by the rearrangement inequality,
\[ \frac{a}{b+c} + \frac{b}{a+c} + \frac{c}{a+b} \geq \frac{a}{a+c} + \frac{b}{a+b} + \frac{c}{b+c} \]
and also
\[ \frac{a}{b+c} + \frac{b}{a+c} + \frac{c}{a+b} \geq \frac{a}{a+b} + \frac{b}{b+c} + \frac{c}{a+c}. \]
Adding up these two inequalities and arranging terms on the RHS conveniently, we conclude

\[ 2 \left( \frac{a}{b+c} + \frac{b}{a+c} + \frac{c}{a+b} \right) \geq \left( \frac{a}{a+c} + \frac{c}{a+b} \right) + \left( \frac{a}{a+c} + \frac{b}{a+b} \right) + \left( \frac{b}{b+c} + \frac{c}{b+c} \right) = 3. \]

This proves the inequality. □

2.2. Use of analysis. To prove an inequality, denote some expression with \( f \) and study this function using analysis.

1. Prove that

\[ (1 + x)^n + (1 - x)^n \leq 2^n \]

for all \( n \geq 1 \) and \( |x| \leq 1 \).

Proof. We may assume \( n \geq 2 \). The function \( f(x) = (1 + x)^n + (1 - x)^n \) has

\[ f''(x) = n(n-1) \left( (1+x)^{n-2} + (1-x)^{n-2} \right) > 0. \]

Since \( f \) is convex, it achieves its maximum on \([-1, 1]\) at one or both of the endpoints. But \( f(-1) = f(1) = 2^n \), which proves the inequality. □

2. Prove that

\[ \frac{x_1}{2-x_1} + \ldots + \frac{x_n}{2-x_n} \geq \frac{n}{2n-1} \]

for any non-negative numbers \( x_1, \ldots, x_n \) so that \( x_1 + \ldots + x_n = 1 \).

Proof. Define the function \( f : D \to \mathbb{R} \) by

\[ f(x_1, \ldots, x_n) = \frac{x_1}{2-x_1} + \ldots + \frac{x_n}{2-x_n} \]

where

\[ D = \{(x_1, \ldots, x_n) : x_i \geq 0, \text{ for all } i\}. \]

We want to find the minimum of \( f \) on \( D \), subject to the constraint \( x_1 + \ldots + x_n = 1 \). Denote

\[ g(x) = x_1 + \ldots + x_n - 1. \]

Assume first that \( f \) achieves its minimum in the interior of \( D \). Then according to the Lagrange multipliers method we have

\[ \nabla f = \lambda \nabla g \]
at a maximum point of \( f \) in \( D \). Note that
\[
\frac{\partial f}{\partial x_i} = \frac{2}{(2 - x_i)^2},
\]
so we need to solve the system
\[
\begin{cases}
\frac{2}{(2 - x_i)} = \lambda & \text{for } i = 1, 2, \ldots, n
\end{cases}
\]
This implies that \( x_1 = x_2 = \ldots = x_n \) at a minimum point. Plugging this into \( g = 0 \) we conclude that \( x_1 = \ldots = x_n = \frac{1}{n} \) at some minimum point in \( D \). But note that
\[
f \left( \frac{1}{n}, \ldots, \frac{1}{n} \right) = \frac{n}{2n - 1},
\]
so in this case the inequality follows.

Let us assume now that \( f \) achieves its minimum on the boundary of \( D \). Note that due to the constraint \( g = 0 \), the minimum of \( f \) cannot occur at infinity. Since \( (x_1, \ldots, x_n) \) is on the boundary of \( D \), it follows that at least one of \( x_i \) is zero. Without loss of generality we may assume \( x_n = 0 \) at the minimum point of \( f \). Note however that
\[
f (x_1, \ldots, x_{n-1}, 0) = \frac{x_1}{2 - x_1} + \ldots + \frac{x_{n-1}}{2 - x_{n-1}},
\]
and \( x_1 + \ldots + x_{n-1} = 1 \).

Either by an induction argument, or by continuing the argument above, we may assume that the inequality we want to prove for \( n \) numbers is true for \( n - 1 \) numbers. So, we may assume that
\[
\frac{x_1}{2 - x_1} + \ldots + \frac{x_{n-1}}{2 - x_{n-1}} \geq \frac{(n - 1)}{2(n - 1) - 1}
\]
for all positive \( x_1, \ldots, x_{n-1} \) so that \( x_1 + \ldots + x_{n-1} = 1 \). Since
\[
\frac{(n - 1)}{2(n - 1) - 1} > \frac{n}{2n - 1}
\]
this proves that in fact the minimum of \( f \) cannot occur on the boundary of \( D \). \( \square \)