INEQUALITIES

1. Classical inequalities

(1) The Cauchy-Schwarz inequality:

$$\left(\sum_{k=1}^{n} a_k b_k\right)^2 \leq \left(\sum_{k=1}^{n} a_k^2\right)\left(\sum_{k=1}^{n} b_k^2\right)$$

and integral version

$$\left(\int_a^b f(x)\, g(x)\, dx\right)^2 \leq \left(\int_a^b f^2(x)\, dx\right)\left(\int_a^b g^2(x)\, dx\right).$$

(2) Hölder inequality:

$$\left|\sum_{k=1}^{n} a_k b_k\right| \leq \left(\sum_{k=1}^{n} |a_k|^p\right)^{\frac{1}{p}} \left(\sum_{k=1}^{n} |b_k|^q\right)^{\frac{1}{q}}$$

and integral version

$$\int_a^b |f(x)\, g(x)|\, dx \leq \left(\int_a^b |f|^p(x)\, dx\right)^{\frac{1}{p}} \left(\int_a^b |g|^q(x)\, dx\right)^{\frac{1}{q}},$$

where $\frac{1}{p}+\frac{1}{q}=1$ and $p, q > 1$.

(3) Minkowski inequality

$$\left(\sum_{k=1}^{n} |a_k+b_k|^p\right)^{\frac{1}{p}} \leq \left(\sum_{k=1}^{n} |a_k|^p\right)^{\frac{1}{p}} + \left(\sum_{k=1}^{n} |b_k|^p\right)^{\frac{1}{p}}$$

and integral version

$$\int_a^b |f(x)+g(x)|^p\, dx \leq \left(\int_a^b |f|^p(x)\, dx\right)^{\frac{1}{p}} + \left(\int_a^b |g|^p(x)\, dx\right)^{\frac{1}{p}}.$$

(4) Jensen inequality:

$$f\left(\frac{x_1+..+x_n}{n}\right)\leq\frac{f\left(x_1\right)+..+f\left(x_n\right)}{n}$$

if f is convex. Integral version

$$f\left(\int_0^1 u\left(x\right)dx\right)\leq\int_0^1 f\left(u\left(x\right)\right)dx,$$

for any continuous function u, if f is convex.

(5) Young's inequality:

$$ab\leq\int_0^a f\left(x\right)dx+\int_0^b f^{-1}\left(x\right)dx,$$

if $f:[0,\infty)\to[0,\infty)$ is continuous, strictly increasing and $f\left(0\right)=0$.

(6) Chebyshev inequality

$$\left(\sum_{k=1}^n a_k\right)\left(\sum_{k=1}^n b_k\right)\leq n\sum_{k=1}^n a_k b_k$$

if $\left(a_k\right)_k$ and $\left(b_k\right)_k$ are increasing sequences. Integral version

$$\left(\int_a^b f\left(x\right)dx\right)\left(\int_a^b g\left(x\right)dx\right)\leq\left(b-a\right)\int_a^b f\left(x\right)g\left(x\right)dx$$

if f and g are both increasing functions.

1.1. **Examples using classical inequalities.** Solve the following.

1. The AM-GM inequality

$$\sqrt[n]{a_1...a_n}\leq\frac{a_1+..+a_n}{n}$$

for all positive numbers $a_1,...,a_n$.

Proof. Use Jensen for $f\left(x\right)=\ln x$, which is concave on $(0,\infty)$. $\square$

2. Prove that

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}$$

for any $a, b > 0$, where $\frac{1}{p} + \frac{1}{q} = 1$ and $p, q > 1$.

Proof. Apply Young's inequality to $f(x) = x^{p-1}$, for $p > 1$. □

3. Let $x_i \in (0, \pi)$ and $x = \frac{x_1 + ... + x_n}{n}$. Prove that

$$\prod_{i=1}^{n} \frac{\sin x_i}{x_i} \leq \left(\frac{\sin x}{x}\right)^n .$$

Proof. Taking log, we see that the statement to prove is equivalent to

$$\frac{1}{n} \sum_{i=1}^{n} \frac{\sin x_i}{x_i} \leq \ln\left(\frac{\sin x}{x}\right). \tag{1}$$

The function $f(t) = \ln\left(\frac{\sin t}{t}\right)$ is concave on $(0, \pi)$, so by Jensen we have

$$\frac{f(x_1) + .. + f(x_n)}{n} \leq f\left(\frac{x_1 + .. + x_n}{n}\right).$$

This implies (1). □

4. Prove that

$$(a_1 ... a_n)^{\frac{1}{n}} + (b_1 ... b_n)^{\frac{1}{n}} \leq ((a_1 + b_1) ... (a_n + b_n))^{\frac{1}{n}}$$

for any positive numbers $a_1, .., a_n$ and $b_1, .., b_n$.

Proof. Scaling trick: the inequality to be proved does not change if we replace a_i by $\lambda_i a_i$ and b_i by $\lambda_i b_i$, where $\lambda_i > 0$.

So we may assume that

$$a_i + b_i = 1, \tag{2}$$

for all $i = 1, 2, ., n$, by simultaneously rescaling a_i and b_i by an appropriate λ_i.

Then the inequality to be proved becomes

$$(a_1 ... a_n)^{\frac{1}{n}} + (b_1 ... b_n)^{\frac{1}{n}} \leq 1.$$

Using the AM-GM inequality we have

$$\begin{aligned} (a_1 .. a_n)^{\frac{1}{n}} &\leq \frac{a_1 + .. + a_n}{n} \\ (b_1 .. b_n)^{\frac{1}{n}} &\leq \frac{b_1 + .. + b_n}{n} \end{aligned}$$

so that adding up we get

$$\begin{aligned} (a_1 ... a_n)^{\frac{1}{n}} + (b_1 ... b_n)^{\frac{1}{n}} &\leq \frac{(a_1 + b_1) + .. + (a_n + b_n)}{n} \\ &= 1 \end{aligned}$$

where the second line follows from (2). □

5. Prove that

$$x^x y^y z^z \geq (xyz)^{\frac{x+y+z}{3}}$$

for any positive numbers $x, y\,, z$.

Proof. Without loss of generality, we may assume that $x \leq y \leq z$. Taking log, we see that the inequality to be proved is equivalent to

$$(x+y+z)(\ln x + \ln y + \ln z) \leq 3\,(x \ln x + y \ln y + z \ln z)\,.$$

Since $x \leq y \leq z$ and $\ln x \leq \ln y \leq \ln z$, the above inequality follows from Chebyshev. □

2. Some techniques of proving inequalities

Not all new inequalities reduce to the classical ones. Sometimes we need new ideas.

2.1. **Rearrangement inequality.**

Lemma 1. *If $a_1 \leq a_2 \leq a_3$ and $b_1 \leq b_2 \leq b_3$ then*

$$a_1 x_1 + a_2 x_2 + a_3 x_3 \leq a_1 b_1 + a_2 b_2 + a_3 b_3$$

for any permutation (x_1, x_2, x_3) of (b_1, b_2, b_3).

The result is true for increasing sequences of any number of terms.

1. Prove that

$$\frac{a}{b+c} + \frac{b}{a+c} + \frac{c}{a+b} \geq \frac{3}{2}$$

for any $a, b, c > 0$.

Proof. Without loss of generality we may assume $a \leq b \leq c$. Then (a, b, c) and $\left(\frac{1}{b+c}, \frac{1}{a+c}, \frac{1}{a+b}\right)$ are similarly arranged, so by the rearrangement inequality,

$$\frac{a}{b+c} + \frac{b}{a+c} + \frac{c}{a+b} \geq \frac{a}{a+c} + \frac{b}{a+b} + \frac{c}{b+c}$$

and also

$$\frac{a}{b+c} + \frac{b}{a+c} + \frac{c}{a+b} \geq \frac{a}{a+b} + \frac{b}{b+c} + \frac{c}{a+c}.$$

Adding up these two inequalities and arranging terms on the RHS conveniently, we conclude

$$\begin{aligned} 2\left(\frac{a}{b+c}+\frac{b}{a+c}+\frac{c}{a+b}\right) &\geq \left(\frac{a}{a+c}+\frac{c}{a+c}\right)+\left(\frac{a}{a+b}+\frac{b}{a+b}\right) \\ &\quad +\left(\frac{b}{b+c}+\frac{c}{b+c}\right) \\ &= 3. \end{aligned}$$

This proves the inequality. □

2.2. **Use of analysis.** To prove an inequality, denote some expression with f and study this function using analysis.

1. Prove that

$$(1+x)^n+(1-x)^n \leq 2^n$$

for all $n \geq 1$ and $|x| \leq 1$.

Proof. We may assume $n \geq 2$. The function $f(x)=(1+x)^n+(1-x)^n$ has

$$\begin{aligned} f''(x) &= n(n-1)\left((1+x)^{n-2}+(1-x)^{n-2}\right) \\ &> 0. \end{aligned}$$

Since f is convex, it achieves its maximum on $[-1,1]$ at one or both of the endpoints. But $f(-1)=f(1)=2^n$, which proves the inequality. □

2. Prove that

$$\frac{x_1}{2-x_1}+..+\frac{x_n}{2-x_n} \geq \frac{n}{2n-1}$$

for any non-negative numbers $x_1,..,x_n$ so that $x_1+...+x_n=1$.

Proof. Define the function $f: D \to \mathbb{R}$ by

$$f(x_1,..,x_n)=\frac{x_1}{2-x_1}+..+\frac{x_n}{2-x_n}$$

where

$$D=\{(x_1,..,x_n) : x_i \geq 0, \text{ for all } i\}.$$

We want to find the minimum of f on D, subject to the constraint $x_1+...+x_n=1$. Denote

$$g(x)=x_1+...+x_n-1.$$

Assume first that f achieves its minimum in the interior of D. Then according to the Lagrange multipliers method we have

$$\nabla f=\lambda \nabla g$$

at a maximum point of f in D. Note that

$$\frac{\partial f}{\partial x_i} = \frac{2}{(2-x_i)^2},$$

so we need to solve the system

$$\left\{ \frac{2}{(2-x_i)^2} = \lambda \text{ for } i = 1, 2, .., n \right.$$

This implies that $x_1 = x_2 = .. = x_n$ at a minimum point. Plugging this into $g = 0$ we conclude that $x_1 = ... = x_n = \frac{1}{n}$ at some minimum point in D. But note that

$$f\left(\frac{1}{n}, .., \frac{1}{n}\right) = \frac{n}{2n-1},$$

so in this case the inequality follows.

Let us assume now that f achieves its minimum on the boundary of D. Note that due to the constraint $g = 0$, the minimum of f cannot occur at infinity. Since $(x_1, .., x_n)$ is on the boundary of D, it follows that at least one of x_i is zero. Without loss of generality we may assume $x_n = 0$ at the minimum point of f. Note however that

$$f(x_1, .., x_{n-1}, 0) = \frac{x_1}{2-x_1} + .. + \frac{x_{n-1}}{2-x_{n-1}},$$

and $x_1 + .. + x_{n-1} = 1$.

Either by an induction argument, or by continuing the argument above, we may assume that the inequality we want to prove for n numbers is true for $n-1$ numbers. So, we may assume that

$$\frac{x_1}{2-x_1} + .. + \frac{x_{n-1}}{2-x_{n-1}} \geq \frac{(n-1)}{2(n-1)-1}$$

for all positive $x_1, .., x_{n-1}$ so that $x_1 + .. + x_{n-1} = 1$. Since

$$\frac{(n-1)}{2(n-1)-1} > \frac{n}{2n-1}$$

this proves that in fact the minimum of f cannot occur on the boundary of D. $\square$