MATH 3974 PROBLEM SEMINAR HOMEWORK

1. Prove that
$$1 - \frac{1}{x} \leq \ln x \leq x - 1,$$
for all $x \geq 1$.

2. If $a_1 + a_2 + ... + a_n = n$, prove that
$$a_1^4 + a_2^4 + ... + a_n^4 \geq n.$$

3. If $x_0 > x_1 > ... > x_n$, prove that
$$x_0 + \frac{1}{x_0 - x_1} + ... + \frac{1}{x_{n-1} - x_n} \geq x_n + 2n.$$

4. Prove that
$$\frac{x}{x + 2y + 2z} + \frac{y}{y + 2x + 2z} + \frac{z}{z + 2x + 2y} \geq \frac{3}{5}$$
for any positive x, y, z.

5. Show that for all a_1,...,$a_n > 0$ we have
$$\left(\sum_{k=1}^{n} (a_k)^3 \right)^2 \leq \left(\sum_{k=1}^{n} (a_k)^2 \right)^3.$$

6. For any $x, y, z \in \mathbb{R}$, prove that
$$2^{x^2} + 2^{y^2} + 2^{z^2} \geq 2^{xy} + 2^{yz} + 2^{xz}.$$

7. Prove that
$$\sqrt{\frac{x}{y+z}} + \sqrt{\frac{y}{x+z}} + \sqrt{\frac{z}{x+y}} > 2$$
for all $x, y, z > 0$.

8. Prove that the positive real numbers a, b, c are the side lengths of a triangle if and only if
$$a^2 + b^2 + c^2 < 2\sqrt{a^2b^2 + b^2c^2 + c^2a^2}.$$