Problem Seminar, Spring 2020-Inequalities in Calculus

1. Prove that any continuously differentiable function $f : [a,b] \to \mathbb{R}$ such that $f(a) = 0$ satisfies:
$$\int_a^b f(x)^2\,\mathrm{d}x \le (b-a)^2 \int_a^b f'(x)^2\,\mathrm{d}x.$$

2. Find
$$\max_{f:[0,3]\to\mathbb{R} \text{ continuous and positive}} \frac{(\int_0^3 f(x)\,\mathrm{d}x)^3}{\int_0^3 f(x)^3\,\mathrm{d}x}.$$

3. Show that
$$\sin x + \tan x > 2x \quad \forall x \in (0, \pi/2).$$

4. (Young's Inequality) Let $f : [0,c] \to \mathbb{R}$ be a strictly increasing and continuous function such that $f(0) = 0$. Then for all $a \in [0,c], b \in [0, f(c)]$:
$$\int_0^a f(x)\,\mathrm{d}x + \int_0^b f^{-1}(x)\,dx \ge ab.$$

5. Let $f : \mathbb{R} \to (0, +\infty)$ be a continuous 1-periodic function. Prove that
$$\int_0^1 \frac{f(x)}{f(x+1/2)}\,dx \ge 1.$$
Hint: Split the integral as $\int_0^1 = \int_0^{1/2} + \int_{1/2}^1$ *and then change variables in the second integral so you would have same limits.*

6. Let $x_i \in \mathbb{R}^+, i = 1,2,3,$ such that $\sum_{i=1}^3 x_i = 3$. Find the minimum value of
$$\frac{x_1^3}{x_1+x_2} + \frac{x_2^3}{x_2+x_3} + \frac{x_3^3}{x_3+x_1}. \tag{1}$$
Hint: Use the discrete version of Hölder's Inequality.