Complex numbers practice problems

Problem 1 (Diff. 1)**.** Let m and n two integers such that each can be expressed as the sum of two perfect squares. Prove that $m \cdot n$ has this property as well. For instance $17 = 4^2 + 1^2$, $13 = 2^2 + 3^2$, and $17 \cdot 13 = 221 = 14^2 + 5^2$.

Problem 2 (Diff. 1)**.** Let $ABCD$ be a convex quadrilateral. Let $M \in [AB]$, let $N \in [BC]$, let $P \in [CD]$, and $Q \in [DA]$ such that

$$\frac{|AM|}{|MB|} = \frac{|DP|}{|PC|} = r \qquad \text{and} \qquad \frac{|BN|}{|NC|} = \frac{|AQ|}{|QD|} = s.$$

Let $\{O\} = MP \cap NQ$. Prove that $\frac{|QO|}{|ON|} = r$ and $\frac{|MO|}{|OP|} = s$.

Hint: Find the complex coordinate of M in terms of those of A and B and of r.

Problem 3 (Diff. 2)**.** Solve $z^{2019} = \bar{z}$.

Problem 4 (Diff. 2)**.** Let $ABCD$ be a convex quadrilateral. Let T and V be points inside the quadrilateral and U, W be points outside such that the angles UAB, TAD, VCB, WCD are all equal, and the angles UBA, VBC, WDC, TDA are all equal. Prove that $UTWV$ is a parallelogram.

Hint: By a small letter, we denote the complex coordinate of the corresponding capital letter. Show that it is enough to prove that $u + w = t + v$. For a fixed $r > 0$ and fixed angle θ, rotating AB around A by θ and then scaling the result by r takes b to $a + z(b - a)$.

Problem 5 (Diff. 3)**.** Consider a regular n-gon which is inscribed in a circle with radius 1. What is the average of the lengths of all $n(n-1)/2$ chords joining different vertices of the n-gon?

Hint: Let $\xi = e^{\pi\imath/2n}$. Note that this is a $2n$-th root of 1, not an n-th root of 1. Prove that $|1 - \xi^{2k}| = \frac{\xi^k - \bar{\xi}^k}{2\imath}$.

Problem 6 (Diff. 3)**.** Find all complex numbers z that verify

$$\big|z - |z+1|\big| = \big|z + |z-1|\big|.$$

Problem 7 (Diff. 4 (Putnam 1991, B2))**.** Suppose f and g are non-constant, differentiable, real-valued functions on $\mathbb{R}$. Furthermore, suppose that for each pair of real numbers x and y

$$f(x+y) = f(x)f(y) - g(x)g(y)$$
$$g(x+y) = f(x)g(y) + g(x)f(y)$$

If $f'(0) = 0$, prove that $f^2(x) + g^2(x) = 1$ for all x.

Problem 8 (Diff. 5)**.** If z is a complex number with $|z| \le 1$, and $\omega = e^{2\pi\imath/3}$, prove that

$$3 \le |z-1| + |z-\omega| + |z-\omega^2| \le 4.$$