Putnam Problem Seminar: Fall 2019-Calculus II

1. (Difficulty 1) Let $f : [0,1] \to \mathbb{R}$ be a continuous function. Prove that

$$\left(\int_0^1 f(x)dx\right)^2 \leq \int_0^1 f^2(x)dx.$$

2. (Difficulty 1) Find

$$\max_{f:[0,3]\to\mathbb{R} \text{ continuous and positive}} \frac{(\int_0^3 f(x))^3 cdx}{\int_0^3 f(x)^3 dx}.$$

3. (Difficulty 1) Let $f : \mathbb{R} \to (0, +\infty)$ be a continuous function. Show that

$$\int_0^1 f(x)dx \geq 2^{\int_0^1 \log_2 f(t)dt}.$$

4. (Difficulty 2) Let $f, g : \mathbb{R} \to \mathbb{R}$ such that f is bounded and not identically equal to zero. Show that if

$$f(x+y) + f(x-y) = 2f(x)g(y)$$

for all $x, y \in \mathbb{R}$ then $|g(y)| \leq 1$ for all $y \in \mathbb{R}$.

5. (Difficulty 2) Find all functions $f : \mathbb{R} \to \mathbb{R}$ which satisfy

$$f(x+y) + f(y+z) + f(z+x) \geq 3f(x+2y+3z)$$

for all $x, y, z \in \mathbb{R}$. *Hint: Using appropriate values for x, y, z compare $f(x)$ with $f(0)$.*