Putnam Problem Seminar: Fall 2019-Calculus I

1. (Difficulty 1) Prove that the series
$$\sum_{n=2}^{\infty} \frac{1}{n \log n}$$
diverges.

2. (Difficulty 1) (Kronecker's Rule) Let p be a polynomial of degree m and let f be a continuous function on some interval $[a, b]$. Let
$$F_1(x) = \int f(x)dx,\ F_2(x) = \int F_1(x)dx, \ldots,\ F_{n+1}(x) = \int F_n(x)dx.$$
Prove that
$$\int_a^b p(x)f(x)dx = \sum_{j=0}^{m} (-1)^j p^{(j)}(x) F_{j+1}(x) \Big|_a^b .$$

3. (Difficulty 2) Evaluate the integral $\int_0^{\pi/2} \frac{1}{1+\tan x} dx$.

4. (Difficulty 3) Let (x_n) be a sequence of real numbers satisfying $x_{n+m} \leq x_n + x_m, n, m \geq 1$. Show that $\lim_{n\to\infty} \frac{x_n}{n}$ exists and is equal to $\inf_{n\in\mathbb{N}} \frac{x_n}{n}$.

5. (Difficulty 4)[Putnam 1997] Let f be a twice-differentiable real-valued function satisfying
$$f(x) + f''(x) = -xg(x)f'(x),$$
where $g(x) \geq 0$ for all real x. Prove that $|f(x)|$ is bounded.