

# Math 3794 Problem Seminar, Homework 5

Due Tuesday, November 5th

**Problem 1 (Difficulty: 1).** Which one of the two numbers

$$\sqrt{4 + \sqrt{7}} - \sqrt{4 - \sqrt{7}} - \sqrt{2} \quad \text{and} \quad 0$$

is larger? Try to avoid using calculators or other electronic devices when solving the problem.

**Problem 2 (Difficulty: 3).** Let  $\{a_n\}_{n=1}^{\infty}$  be the sequence defined in the following way

$$a_1 = 1, \quad a_k = a_{k-1} + \frac{1}{a_{k-1}}, \quad k = 2, 3, 4, \dots$$

Show that  $14 < a_{100} < 18$ .

**Problem 3 (Difficulty: 1).** Let  $m$  be a positive integer and let  $a_1, a_2, \dots, a_m$  be  $m$  real numbers. Prove that

$$(a_1 + a_2 + \dots + a_m)^2 \leq m(a_1^2 + a_2^2 + \dots + a_m^2).$$

Under what conditions on  $a_1, a_2, \dots, a_m$  does the equality hold?

**Problem 4 (Difficulty: 2; a half of an old B-6 Putnam question).** Let

$$H_n = 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n}, \quad n = 1, 2, 3, \dots$$

Show that

$$n(n+1)^{\frac{1}{n}} < n + H_n, \quad n = 1, 2, 3, \dots$$

**Problem 5 (Difficulty: 4; A-1 Putnam question).** Let  $a_1, a_2, \dots, a_n$  and  $b_1, b_2, \dots, b_n$  be nonnegative real numbers. Show that

$$(a_1 a_2 \dots a_n)^{1/n} + (b_1 b_2 \dots b_n)^{1/n} \leq [(a_1 + b_1)(a_2 + b_2) \dots (a_n + b_n)]^{1/n}.$$