Math 3974 Problem Seminar Homework 1

Due September 24, 2019

Problem 1.1. Let $a_1 = 3$, $a_2 = 6$, $a_3 = 11$ be the first three terms of a sequence. Assume that the sequence satisfies the following iteration relation.

$$a_{n+2} = 3a_{n+1} - 2a_n - 1, \quad \text{for } n \geq 1.$$

(1) (deg 1) Compute a_3 to a_8 (and more terms if needed) (No need to write out the computation.)

(2) (deg 1) Think about what is the asymptotic behavior of such a sequence, and make a guess of a closed formula for a_n.

(3) (deg 1) Prove your guess by induction.

Problem 1.2. Let $a_1 = a_2 = 1$, $a_3 = 2$ be the first three terms of a sequence. Assume that the sequence satisfies the following iteration relation

$$\det \begin{pmatrix} a_n & a_{n+1} \\ a_{n-1} & a_n \end{pmatrix} = (-1)^{n+1}.$$

(1) (deg 1) Compute the first several terms of the series and make a guess of which famous series this is.

(2) (deg 3) Prove that this sequence is equal to this famous sequence. (For this, you need to verify that a_n also satisfies a simpler iteration relation. You may use induction, by assuming that the simpler iteration holds for smaller a_n's and then prove it for the next a_n.)

Problem 1.3. [2008-B2] Let $F_0(x) = \ln x$. For $n \geq 0$ and $x > 0$, let $F_{n+1}(x) = \int_0^x F_n(t)dt$. Evaluate

$$\lim_{n\to\infty} \frac{n! F_n(1)}{\ln n}.$$

We suggest to follow the steps below.

(1) (deg 1) Compute by hand $F_1(x)$, $2F_2(x)$, and verify that $6F_3(x) = x^3 \ln x - \frac{11}{6}x^3$.

(2) (deg 2) Compute more terms of $n!F_n(x)$ and look at the difference of $n!F_n(x)$ and $(n-1)!f_{n-1}(x)$. Make a guess of the general formula of $F_n(x)$.

(3) (deg 1) Prove the general formula of $F_n(x)$ by induction.

(4) (deg 1) Complete the computation of the limit in this problem.

Problem 1.4 (2002-A1)**.** Let k be a fixed positive integer. The n-th derivative of $\frac{1}{x^k-1}$ has the form $\frac{P_n(x)}{(x^k-1)^{n+1}}$ where $P_n(x)$ is a polynomial. Find $P_n(1)$.

(1) (deg 1) Get an iterative relation between $P_n(x)$ and $P_{n+1}(x)$. (Just take the derivative.)

(2) (deg 2) Evaluate the iterative relation at $x = 1$. Solve the problem.

Problem 1.5 (1997-B1)**.** Let $\{x\}$ denote the distance between the real number x and the nearest integer. For each positive integer n, evaluate

$$\sum_{m=1}^{6n-1} \min\left(\left\{\frac{m}{6n}\right\}, \left\{\frac{m}{3n}\right\}\right).$$

(1) (deg 1) Compute this for $n = 1, 2, 3$ (and maybe more if needed), write out each term in the sum.

(2) (deg 4) Stare at the sum computed in (1), and make guesses on what the general form should look like. Prove your guesses.