Complex numbers

1 Overview

Definition 1. A *complex number* is a pair $(a, b) \in \mathbb{R}^2$ denoted $\overline{z} = a + b\imath$. We also denote $a = \Re z$ and $b = \Im z$. The set of complex numbers is denoted $\mathbb{C}$.

We add complex numbers componentwise

$$(a, b) + (a', b') = (a + a', b + b').$$

We multiply complex numbers by the weird rule

$$(a, b) \cdot (a', b') = (aa' - bb', ab' + a'b),$$

corresponding to $(a + b\imath)(a' + b'\imath) = aa' + ab'\imath + ba'\imath - bb'$. In particular, if we put $0 = (0, 0)$, and $1 = (1, 0)$ and $\imath = (0, 1)$, then

$$\begin{aligned} z + 0 = 0 + z = z \ \forall z \in \mathbb{C} \\ z \cdot 1 = 1 \cdot z = z \ \forall z \in \mathbb{C} \\ \imath^2 = -1 \end{aligned}$$

With these operations, $\mathbb{C}$ forms a field: we can add, subtract, multiply, and divide by nonzero elements. 0 and 1 are identity elements for addition and multiplication respectively.

Two complex numbers are equal iff their real parts and imaginary parts are equal.

The *complex conjugate* of $z = a + b\imath$ is $\bar{z} := a - b\imath$.
The *length* or *modulus* of a complex number is

$$|z| := \sqrt{z \cdot \bar{z}} = \sqrt{a^2 + b^2}.$$

We have $z = 0$ iff $|z| = 0$.

A complex number of form $z = a + 0 \cdot \imath$ with $a \in \mathbb{R}$ is called *real*. A complex number $z = b\imath$ with $b \in \mathbb{R}$ is called *imaginary*.
z is real iff $z = \bar{z}$, and imaginary iff $z = -\bar{z}$.

We can also represent complex numbers in *polar coordinates*. Usually we identify a complex number by its real and imaginary part, looking at them as cartesian coordinates. We can also identify $z \in \mathbb{C}$ by its length $r = |z|$ and by the angle θ between the line through z in $\mathbb{C}$ and the real axis, measured counterclockwise from the real axis. Then we have

$$z = r(\cos\theta + \imath \sin\theta).$$

From Euler's formula

$$e^{i\pi} = -1 \text{ generalized to } e^{i\theta} = \cos\theta + \imath \sin\theta,$$

we also write

$$z = re^{\imath\theta}.$$

We call θ the *argument* of z. It is only unique modulo 2π. We have

$$\bar{z} = re^{-\imath\theta}.$$

The inverse of a nonzero complex number $z = a + b\imath = re^{\imath\theta}$ (so $r > 0$ and a, b not both 0) is

$$\frac{1}{z} = \frac{\bar{z}}{|z|^2} = \frac{a}{a^2+b^2} - \frac{b}{a^2+b^2}\imath = \frac{1}{r}e^{-\imath\theta}.$$

Theorem 2 ($\mathbb{C}$ is algebraically closed)**.** *Any polynomial $p(x) = a_n x^n + \ldots + a_0 \in \mathbb{C}[x]$ (or $\mathbb{R}[x]$) with $a_n \neq 0$ decomposes uniquely up to order of factors as*

$$p(x) = a_n(x - z_1) \cdot \ldots \cdot (x - z_n).$$

This means that $p(x)$ has exactly n complex roots when counted with multiplicity.

Example 3 (Roots of unity)**.** . Let $n \geq 1$ be an integer. The n-th roots of unity are the complex roots of $x^n - 1 = 0$. They are

$$\mu_n := \{1,\ e^{\frac{2\pi\imath}{n}},\ e^{\frac{4\pi\imath}{n}},\ \ldots,\ e^{\frac{2(n-1)\pi\imath}{n}}\}.$$

All roots of unity have length 1. In $\mathbb{R}^2$ they form a regular n-gon inscribed in the unit circle.

If ω is an n-th root of unity, then $\omega^{m+n} = \omega^m$ for any integer m.

The product of two n-th roots of unity is again an n-th root of unity. More generally, the product of an n-th root of unity and an m-th root of unity is an $\operatorname{lcm}(n, m)$-th root of unity.

We have $\mu_n \subseteq \mu_m$ iff $n \mid m$.

An n-th root of unity $\omega = e^{2k\pi\imath/n}$ is called *primitive* if $\gcd(k, n) = 1$. The primitive roots have the property that any other n-th root of unity is some power of ω. This can be proved with the Euclidean algorithm.

Let $\omega = e^{2\pi\imath/n}$ and consider the *cyclotomic* polynomial

$$\Phi_n(x) := \prod_{\substack{\gcd(k,n)=1 \\ 1 \leq k \leq n-1}} (x - \omega^k).$$

It turns out that $\Phi_n(x)$ is a monic with integer coefficients, and irreducible in $\mathbb{Q}[x]$.

It is the monic polynomial of smallest possible degree with integer coefficients that has ω as root. Its roots are all the primitive n-th roots of unity. It follows that

$$x^n - 1 = \prod_{d|n} \Phi_d(x)$$

From Möbius inversion, we also get

$$\Phi_n(x) = \prod_{d|n} (x^d - 1)^{\mu(n/d)},$$

where

$$\mu(n) := \begin{cases} 0 & \text{, if } n \text{ is not squarefree} \\ 1 & \text{, if } n = 1 \\ (-1)^k & \text{, if } n \text{ is product of } k \text{ distinct primes} \end{cases}$$

From these we find

$$\begin{aligned}
\Phi_1(x) &= x - 1 \\
\Phi_2(x) &= x + 1 \\
\Phi_3(x) &= x^2 + x + 1 \\
\Phi_4(x) &= x^2 + 1 \\
\Phi_5(x) &= x^4 + x^3 + x^2 + x + 1 \\
\Phi_6(x) &= x^2 - x + 1 \\
\Phi_7(x) &= x^6 + x^5 + x^4 + x^3 + x^2 + x + 1 \\
\Phi_8(x) &= x^4 + 1
\end{aligned}$$

The degree of $\Phi_n(x)$ is the number of primitive n-th roots of unity, which is the number of multiplicatively invertible residues modulo n, which is Euler's number

$$\varphi(n) = n \cdot \prod_{p|n} \left(1 - \frac{1}{p}\right),$$

where p ranges through the prime divisors of n.

Problem 4 (Putnam 1991, B2)**.** Suppose f and g are non-constant, differentiable, real-valued functions on $\mathbb{R}$. Furthermore, suppose that for each pair of real numbers x and y

$$\begin{aligned}
f(x+y) &= f(x)f(y) - g(x)g(y) \\
g(x+y) &= f(x)g(y) + g(x)f(y)
\end{aligned}$$

If $f'(0) = 0$, prove that $f^2(x) + g^2(x) = 1$ for all x.

Problem 5. Solve $z^5 + z + 1 = 0$. At least find 2 roots.

There are formulas for solving degree ≤ 4 equations though they are not pleasant.

There are no general formulas for arbitrary equations of degree 5 or higher.

Problem 6. Find closed formulas for the following

(i) $\sum_{k=0}^{n} \binom{n}{k}$.

(ii) $\sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k}$.

(iii) $\sum_{k=0}^{\lfloor n/3 \rfloor} \binom{n}{3k}$.

Problem 7. Consider a regular n-gon which is inscribed in a circle with radius 1. What is the product of the lengths of all $n(n-1)/2$ diagonals of the polygon (this includes the sides of the n-gon)?

2 Complex numbers and Euclidean Geometry

Problem 8. Let $ABCD$ be a convex quadrilateral in the plane. Denote by a, b, c, d the complex coordinates of the vertices after identifying $\mathbb{R}^2 \approx \mathbb{C}$. Prove that $ABCD$ is a parallelogram if and only if $a + c = b + d$.

Problem 9. Let AB be a segment in the plane with A, B having complex coordinates, a, b. Let $r > 0$ be a positive real, and let θ be an angle. What are the complex coordinates of the point C in the plane obtained by rotating AB by the angle θ around the point A, and then scaling the resulting segment by r?

Problem 10. Let $ABCD$ be a convex quadrilateral. Let T and V be points inside the quadrilateral and U, W be points outside such that the angles UAB, TAD, VCB, WCD are all equal, and the angles UBA, VBC, WDC, TDA are all equal. Prove that $UTWV$ is a parallelogram.

Problem 11. Let A, B, C be distinct points in the plane and a, b, c be their coordinates in $\mathbb{C}$. Prove that A, B, C are collinear if and only if

$$\frac{c-a}{b-a} = \frac{\bar{c}-\bar{a}}{\bar{b}-\bar{a}}.$$

Problem 12. Let $ABCD$ be a convex quadrilateral. Let $M \in [AB]$, let $N \in [BC]$, let $P \in [CD]$, and $Q \in [DA]$ such that

$$\frac{AM}{MB} = \frac{DP}{PC} = r \qquad \text{and} \qquad \frac{BN}{NC} = \frac{AQ}{QD} = s.$$

Let $\{O\} = MP \cap NQ$. Prove that $\frac{QO}{ON} = r$ and $\frac{MO}{OP} = s$.

Problem 13. Let A, B, C, D be points in the plane. Prove that $AC \perp BD$ if and only if $\frac{d-b}{c-a} = -\frac{\bar{d}-\bar{b}}{\bar{c}-\bar{a}}$.