SOLUTIONS TO 2019 STUART SIDNEY CALCULUS COMPETITION

Tuesday 26 March, 2018, 6:30–8:00 p.m.

(1) Let $f(x)$ be a function that is odd and differentiable on $(-\infty, +\infty)$.
  (a) Prove that its derivative $f'(x)$ is an even function.
  (b) Is the converse statement true?

Solution. (a) Since $f(x)$ is an odd function, $f(x) = -f(-x)$. Taking derivative of this equality on both sides with respect to x, we get

$$f'(x) = -(f(-x))' = -(-f'(-x)) = f'(-x).$$

Note that the second equality uses the chain rule.

(b) The converse statement is not true, as an odd function requires that $f(0) = 0$, yet only knowing $f'(x)$ is an even function will not give that information. For example, $f(x) = x + 1$ is not an odd function, yet $f'(x) = 1$ is an even function.

(2) Recall the equality from geometric series:

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots, \qquad \text{for } |x| < 1.$$

  (a) Compute the limit of the power series

$$1 \cdot 2 + (2 \cdot 3)x + (3 \cdot 4)x^2 + (4 \cdot 5)x^3 + (5 \cdot 6)x^4 + \cdots \qquad |x| < 1.$$

  as a rational function in x;
  (b) Compute

$$1 - \frac{1 \cdot 2}{2} + \frac{2 \cdot 3}{2^2} - \frac{3 \cdot 4}{2^3} + \frac{4 \cdot 5}{2^4} - \cdots$$

Solution. Starting with the given geometric series, we take the derivatives with respect to x (this is okay as the infinite sum is absolutely convergent):

$$\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + 4x^3 + \cdots$$

The right hand side is absolutely convergent again when $|x| < 1$ by the ratio test for example. Taking the derivatives again, we get

$$\frac{2}{(1-x)^3} = 2 + (2 \cdot 3)x + (3 \cdot 4)x^2 + (4 \cdot 5)x^3 + \cdots$$

So the answer to (a) is $\dfrac{2}{(1-x)^3}$.

(b) Evaluating the above equality at $x = -\frac{1}{2}$, we get

$$\frac{2}{(1+\frac{1}{2})^3} = 1 \cdot 2 - \frac{2 \cdot 3}{2} + \frac{3 \cdot 4}{2^2} - \frac{4 \cdot 5}{2^3} + \cdots$$

Dividing both sides by -2 and add 1, one get

$$1 - \frac{1}{(1+\frac{1}{2})^3} = 1 - \frac{1 \cdot 2}{2} + \frac{2 \cdot 3}{2^2} - \frac{3 \cdot 4}{2^3} + \frac{4 \cdot 5}{2^4} - \cdots$$

The answer to (b) is the left hand side above, namely $1 - \dfrac{2^3}{3^3} = \dfrac{19}{27}$.

(3) Construct one polynomial $f(x)$ with real coefficients and with all of the following properties:
(a) $f(x)$ is an even function, in other words $f(x) = f(-x)$;
(b) $f(2) = f(-2) = 0$,
(c) $f(x) > 0$ when $-2 < x < 2$, and
(d) the maximum of $f(x)$ is achieved at $x = 1$ and $x = -1$.
Justify your answer.

Solution. We start with a function $g(x)$ that achieves maximum at $x = \pm 1$ and minimal at $x = 0$; for this, one may take $g(x)$ such that $g'(x)$ has zero at $x = \pm 1, 0$, e.g. $g'(x) = x - x^3$. Integrating, we may take $g(x) = \frac{x^2}{2} - \frac{x^4}{4}$. To modify it so that it satisfies conditions (b) and (c), we take

$$f(x) = g(x) - g(2) = \frac{x^2}{2} - \frac{x^4}{4} + 2.$$

(4) Consider the first quadrant quarter unit disk

$$QD = \left\{(x, y) \;\mid\; x \geq 0,\; y \geq 0,\; x^2 + y^2 \leq 1\right\}.$$

Assuming uniform density, find the coordinates of the center of mass of QD. (Hint: the equality $\sin 3x = 3 \sin x - 4 \sin^3 x$ might be helpful.)

Solution. We compute the integral

$$\begin{aligned}
\int_0^1 x\sqrt{1-x^2}dx &\overset{x=\sin t}{=} \int_0^{\pi/2} \sin t \cos t (\cos t dt) \\
&= \int_0^{\pi/2} \sin t - \sin^3 t dt \\
&= \int_0^{\pi/2} \frac{1}{4}(\sin t + \sin 3t) dt \\
&= -\frac{1}{4} \cos t \Big|_0^{\pi/2} - \frac{1}{12} \cos 3t \Big|_0^{\pi/2} \\
&= -0 + \frac{1}{4} - 0 + \frac{1}{12} = \frac{1}{3}.
\end{aligned}$$

Then the x- and y-coordinates of the center of mass of QD are

$$\frac{\frac{1}{3}}{\frac{\pi}{4}} = \frac{4}{3\pi}.$$

(5) Which one of the numbers

$$\int_0^\pi e^{\sin^2 x} dx \quad \text{and} \quad \frac{3\pi}{2}$$

is larger? Justify your answer.

Solution Note that

$$e^{\sin^2 x} = 1 + \sin^2 x + \frac{\sin^4 x}{2} + \cdots \geq 1 + \sin^2 x.$$

This inequality is strict on the interval $(0, \pi)$. So we must have

$$\int_0^\pi e^{\sin^2 x} dx > \int_0^\pi (1 + \sin^2 x) dx = \pi + \int_0^\pi \sin^2 x dx.$$

Yet $\sin^2 x = \frac{1-\cos 2x}{2}$. So

$$\int_0^\pi \sin^2 x dx = \int_0^\pi \frac{1 - \cos 2x}{2} dx = \frac{\pi}{2} - \frac{1}{4} \sin 2x \Big|_0^\pi = \frac{\pi}{2}.$$

Combining these two lines, we deduce that

$$\int_0^\pi e^{\sin^2 x} dx > \pi + \frac{\pi}{2} = \frac{3\pi}{2}.$$

(6) Let $f : \mathbb{R} \to \mathbb{R}$ be a periodic continuous function, of period $T > 0$, that is $f(x+T) = f(x)$ for any $x \in \mathbb{R}$. Prove that

$$\lim_{x\to\infty} \frac{1}{x} \int_0^x f(t)\, dt = \frac{1}{T} \int_0^T f(t)\, dt.$$

Solution. For any $x > 0$, there exists a unique positive integer $k > 0$ and $0 \le a_k < T$ so that

$$x = kT + a_k.$$

Since $x \to \infty$, then $k \to \infty$. We have,

$$\begin{aligned} \frac{1}{x} \int_0^x f(t)\, dt &= \frac{1}{kT + a_k} \int_0^{kT+a_k} f(t)\, dt \\ &= \frac{1}{kT + a_k} \left(\int_0^{kT} f(t)\, dt + \int_{kT}^{kT+a_k} f(t)\, dt \right). \end{aligned}$$

Since f is periodic of period T, we know that

$$\begin{aligned} \int_0^{kT} f(t)\, dt &= k \int_0^T f(t)\, dt \\ \int_{kT}^{kT+a_k} f(t)\, dt &= \int_0^{a_k} f(t)\, dt. \end{aligned}$$

Therefore, we get

$$\frac{1}{x} \int_0^x f(t)\, dt = \frac{k}{kT + a_k} \int_0^T f(t)\, dt + \frac{1}{kT + a_k} \int_0^{a_k} f(t)\, dt.$$

Making $k \to \infty$ we obtain the result.

(7) Suppose $(a_n)_{n\ge1}$ is a *decreasing sequence with positive terms* such that

$$\sum_{n=1}^{\infty} a_n < \infty.$$

Prove that:

(a) The sequence $x_n = (a_1 + a_2 + \cdots + a_n) - na_n$ is bounded and increasing.
(b) The sequence $(na_n)_{n\ge1}$ converges to zero when n goes to infinite.

Solution. (a) The sequence x_n is bounded because $a_n > 0$, so

$$x_n \leq \sum_{n=1}^{\infty} a_n < \infty.$$

The sequence is increasing because

$$x_{n+1} - x_n = n\left(a_n - a_{n+1}\right) > 0.$$

(b) Since the sequence $(x_n)_{n\geq 1}$ is bounded and increasing, we know that it has a limit. Therefore, the limit

$$\lim_{n\to\infty} na_n = \lim_{n\to\infty} \Big((a_1 + \cdots + a_n) \; - \; x_n\Big) = \lim_{n\to\infty} (a_1 + \cdots + a_n) - \lim_{n\to\infty} x_n$$

exists. Write L for this limit, and we shall show that $L = 0$.

Suppose $L > 0$. Then by comparison test, $\sum a_n$ and $\sum \frac{1}{n}$ either simultaneously converge or simultaneously diverge. Yet we know $\sum \frac{1}{n}$ diverges and $\sum a_n$ converges. This is a contradiction.

In conclusion, $L = 0$. □

(8) Find all absolute minimum points for the function $f(x, y) = x^4 + y^4 - 4xy$, where $x, y \in \mathbb{R}$.

Solution. To find critical points we solve

$$\begin{aligned} \frac{\partial f}{\partial x} &= 0 \\ \frac{\partial f}{\partial y} &= 0. \end{aligned}$$

It follows that

$$\begin{aligned} 4x^3 - 4y &= 0 \\ 4y^3 - 4x &= 0. \end{aligned}$$

From here we get

$$x^3 = y \quad \text{and } x = y^3.$$

Then we obtain that $x^9 = x$, which has solutions $x = -1$, $x = 0$ and $x = 1$. The points $A(-1, -1)$, $O(0, 0)$ and $B(1, 1)$ are critical points for f. Since $f(0, 0) = 0$ and $f(-1, -1) = f(1, 1) = -2$, only $A(-1, -1)$ and $B(1, 1)$ can be absolute minimum. This is indeed the case, because

$$\begin{aligned} f(x, y) + 2 &= x^4 + y^4 - 4xy + 2 \\ &= (x^2 - y^2)^2 + 2x^2y^2 - 4xy + 2 \\ &= (x^2 - y^2)^2 + 2(xy - 1)^2 \geq 0. \end{aligned}$$

Consequently, $f(x, y) \geq -2$, for all $x, y \in \mathbb{R}$. This proves that A and B are absolute minimum points.

(9) Compute

$$\iiint\limits_S \frac{dxdydz}{(1 + x + y + z)^2}$$

where $S = \{x \geq 0, y \geq 0, z \geq 0, x + y + z \leq 1\}$.

Solution. The solid S is the interior of the tetrahedron defined by $x = 0, y = 0, z = 0$ and $x + y + z = 1$. This can be written as $0 \le z \le 1 - (x + y)$, where $(x, y) \in D$, for $D = \{x \ge 0, y \ge 0, x + y \le 1\}$. Hence

$$
\begin{aligned}
\iiint\limits_S \frac{dxdydz}{(1+x+y+z)^2} &= \iint\limits_D \left(\int_0^{1-(x+y)} \frac{dz}{(1+x+y+z)^2} \right) dxdy \\
&= \iint\limits_D \left(-\frac{1}{1+x+y+z} \right) \Bigg|_0^{1-(x+y)} dxdy \\
&= \iint\limits_D \left(\frac{1}{1+x+y} - \frac{1}{2} \right) dxdy \\
&= \int_0^1 \int_0^{1-x} \left(\frac{1}{1+x+y} - \frac{1}{2} \right) dydx \\
&= \int_0^1 \left(\ln(1+x+y) - \frac{1}{2} y \right) \Bigg|_0^{1-x} dx \\
&= \int_0^1 \left(\ln 2 - \ln(1+x) - \frac{1}{2}(1-x) \right) dx \\
&= \left((\ln 2)\, x - (1+x)\ln(1+x) + \frac{1}{2} x + \frac{1}{4} x^2 \right) \Bigg|_0^1 \\
&= -\ln 2 + \frac{3}{4}.
\end{aligned}
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