LINEAR ALGEBRA SET 2, DUE MAY 6TH

1 (2 pts). Let a, b, c, d be real numbers such that $c \neq 0$ and $ad - bc = 1$. Prove that there exist numbers u, v so that

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix} \begin{pmatrix} 1 & v \\ 0 & 1 \end{pmatrix}.$$

2 (3 pts). Let A, B be $n \times n$ matrices so that

$$tr\left(AA^T + BB^T\right) = tr\left(AB + A^T B^T\right).$$

(a) For $X = A - B^T$, compute $tr\left(XX^T\right)$.
(b) Prove that $A = B^T$.

3 (3 pts). Prove that for any 2×2 matrices $A_1, ..., A_n$ we have

$$\sum_{\varepsilon_1, ... \varepsilon_n} \det\left(\varepsilon_1 A_1 + \varepsilon_2 A_2 + ... + \varepsilon_n A_n\right) = 2^n \left(\det A_1 + \det A_2 + ... + \det A_n\right),$$

where the sum is taken over $\varepsilon_1, ..., \varepsilon_n \in \{-1, 1\}$.

4 (2 pts). Prove that for any 3×3 matrices A and B we have the identity

$$\det(AB - BA) = \frac{1}{3} tr\left((AB - BA)^3\right).$$

5 (2 pts). Let A, B be $n \times n$ matrices with real entries so that $AB = BA$. Prove that

$$\det\left(A^2 + B^2\right) \geq 0.$$

6 (3 pts). Prove that if A, B are 2×2 matrices with real entries, then
(a) we have

$$\begin{aligned} & \det\left(A^2 + B^2\right) + \det(AB + BA) \\ = & \frac{1}{2}\det(A + B)^2 + \frac{1}{2}\det(A - B)^2. \end{aligned}$$

(b) we have

$$\det\left(A^2 + B^2\right) + \det(AB + BA) \geq 0.$$

7 (3 pts). Let A, B be $n \times n$ matrices so that $3AB - 2BA = I_n$. Prove that

$$\det(AB - BA) = 0.$$

8 (2 pts). Calculate

$$\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}^n.$$