Complex numbers homework
Due April 29

**Problem 1** (Diff. 1). Let $m$ and $n$ two integers such that each can be expressed as the sum of two perfect squares. Prove that $m \cdot n$ has this property as well. For instance $17 = 4^2 + 1^2$, $13 = 2^2 + 3^2$, and $17 \cdot 13 = 221 = 14^2 + 5^2$.

**Problem 2** (Diff. 1). Let $ABCD$ be a convex quadrilateral. Let $M \in [AB]$, let $N \in [BC]$, let $P \in [CD]$, and $Q \in [DA]$ such that $|AM| = |MB| = r$ and $|BN| = |NC| = s$. Let $\{O\} = MP \cap NQ$. Prove that $|QO| = |ON|$ and $|MO| = |OP| = s$.

**Hint:** Find the complex coordinate of $M$ in terms of those of $A$ and $B$ and of $r$.

**Problem 3** (Diff. 2). Solve $z^{2019} = \bar{z}$.

**Problem 4** (Diff. 2). Let $ABCD$ be a convex quadrilateral. Let $T$ and $V$ be points inside the quadrilateral and $U,W$ be points outside such that the angles $UAB$, $TAD$, $VCB$, $WCD$ are all equal, and the angles $UBA$, $VBC$, $WDC$, $TDA$ are all equal. Prove that $UTWV$ is a parallelogram.

**Hint:** By a small letter, we denote the complex coordinate of the corresponding capital letter. Show that it is enough to prove that $u + w = t + v$. For a fixed $r > 0$ and fixed angle $\theta$, rotating $AB$ around $A$ by $\theta$ and then scaling the result by $r$ takes $b$ to $a + z(b - a)$.

**Problem 5** (Diff. 3). Consider a regular $n$-gon which is inscribed in a circle with radius 1. What is the average of the lengths of all $(n(n - 1)/2)$ chords joining different vertices of the $n$-gon?

**Hint:** Let $\xi = e^{\pi i/2n}$. Note that this is a $2n$-th root of 1, not an $n$-th root of 1. Prove that $|1 - \xi^{2k}| = \frac{\xi^k - \bar{\xi}^k}{2i}$.

**Problem 6** (Diff. 3). Find all complex numbers $z$ that verify $|z - |z + 1|| = |z + |z - 1||$.

**Problem 7** (Diff. 4 (Putnam 1991, B2)). Suppose $f$ and $g$ are non-constant, differentiable, real-valued functions on $\mathbb{R}$. Furthermore, suppose that for each pair of real numbers $x$ and $y$

\[
\begin{align*}
f(x + y) &= f(x)f(y) - g(x)g(y) \\
g(x + y) &= f(x)g(y) + g(x)f(y)
\end{align*}
\]

If $f'(0) = 0$, prove that $f^2(x) + g^2(x) = 1$ for all $x$.

**Problem 8** (Diff. 5). If $z$ is a complex number with $|z| \leq 1$, and $\omega = e^{2\pi i/3}$, prove that $3 \leq |z - 1| + |z - \omega| + |z - \omega^2| \leq 4$. 

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