# Probabilities

## 1 Roll the dice

**Problem 1.** You roll a standard die three times. The first two times it showed 6. What is the probability that it shows 6 the third time as well?

**Problem 2.** When you play the standard version of Settlers of Catan, you roll a pair of dice simultaneously, and consider the outcome as the sum of the numbers on the top faces. This is a number between 2 and 12. What is the most common outcome?

## 2 Monty Hall

**Problem 3.** Monty Hall was the game host of "Let's make a deal". On the show, the big prize would be hidden behind one of 3 doors, while behind the other two there would be something of no value. The contestant chooses one door, then Monty, who knows where the prize is, opens one of other two doors where he knows that the prize is not hiding. The contestant is then given a choice. Should he keep his first choice, or switch to the other remaining closed door? What is the probability that the prize is behind the other door?

## 3 Geometry

**Problem 4.** Break a line segment of length $\ell$ into three parts at random points. Prove that the probability that the three resulting segments can form the sides of a triangle is $\frac{1}{4}$.

**Problem 5.** Three points are chosen randomly in the plane. What is the probability that the triangle that they form is acute? Give a meaning to this problem, then solve it.

**Problem 6** (Buffon's needle)**.** Parallel lines are drawn on the plane, equally spaced at distance $d$. Let $l < d$ and pick a straight infinitesimally thin stick of length $l$. Through the stick randomly on the plane. Show that the probability of it crossing a line is $\frac{2l}{\pi d}$.

## 4 Bayes's Theorem

Let $A$ and $B$ be events. Denote $P(A)$ the probability that $A$ happens. Denote $P(A|B)$ the probability that $A$ happens given that $B$ already happened. Then

$$P(A \cap B) = P(B)P(A|B).$$

Bayes' Theorem is the corollary

$$P(B|A) = \frac{P(A|B)P(B)}{P(A)}.$$

**Example 7.** Marie is getting married tomorrow, at an outdoor ceremony in the desert. In recent years, it has rained only 5 days each year. Unfortunately, the weatherman has predicted rain for tomorrow. When it actually rains, the weatherman correctly forecasts rain 90% of the time. When it doesn't rain, he incorrectly forecasts rain 10% of the time. What is the probability that it will rain on the day of Marie's wedding?

Solution: The sample space is defined by two mutually-exclusive events - it rains or it does not rain. Additionally, a third event occurs when the weatherman predicts rain. Notation for these events appears below.

Event A1. It rains on Marie's wedding. Event A2. It does not rain on Marie's wedding. Event B. The weatherman predicts rain. In terms of probabilities, we know the following:

$$P(A1) = \frac{5}{365} = 0.0136985 \qquad \text{[It rains 5 days out of the year.]}$$
$$P(A2) = \frac{360}{365} = 0.9863014 \qquad \text{[It does not rain 360 days out of the year.]}$$
$$P(B|A1) = 0.9 \qquad \text{[When it rains, the weatherman predicts rain 90\% of the time.]}$$
$$P(B|A2) = 0.1 \qquad \text{[When it does not rain, the weatherman predicts rain 10\% of the time.]}$$

We want to know $P(A1|B)$, the probability it will rain on the day of Marie's wedding, given a forecast for rain by the weatherman. The answer can be determined from Bayes' theorem, as shown below.

$$\begin{aligned} P(A1|B) &= \frac{P(A1)P(B|A1)}{P(A1)P(B|A1) + P(A2)P(B|A2)} \\ &= \frac{(0.014)(0.9)}{[(0.014)(0.9) + (0.986)(0.1)]} \\ &= 0.111 \end{aligned}$$

Note the somewhat unintuitive result. Even when the weatherman predicts rain, it rains only about 11% of the time. Despite the weatherman's gloomy prediction, there is a good chance that Marie will not get rained on at her wedding.

**Problem 8.** A drug test returns positive for 99% of actual drug users, and returns 99% negative for non-users. For an arbitrarily selected person, what is the probability that if the test returned positive, then he is actually a user? What if the test returns positive for 100% of users?

# 5 Combinatorics

**Problem 9** (2007-A-3)**.** Let $k$ be a positive integer. Suppose that the integers $1, 2, 3, \ldots, 3k + 1$ are written down in random order. What is the probability that at no time during the process, the sum of the integers that have been written up to that time is a positive integer divisible by 3? Your answer should be in closed form, but may include factorials