Math 3974 Problem Seminar Homework 5

Due Monday, March 25th

Problem 1 (3pts). Let $\{a_n\}$ be a sequence of real numbers satisfying

$$a_n = \frac{1}{1-a_{n+1}} - \frac{1}{1+a_{n+1}}, \quad n = 1, 2, 3, 4, \dots.$$

Find $\lim_{n\to\infty} a_n$.

Problem 2 (2pts). Let $\{a_n\}$ be a sequence of real numbers defined by $a_1 = 2$ and

$$a_{n+1} = \frac{2a_n}{a_n^2 - 1}.$$

Prove that the sequence does not terminate, that is, a_n is well defined for all positive integers n.

Problem 3 (4pts). Give an example of a bounded sequence $\{a_n\}$ that does not converge but for which we have

$$(a_{n+1} - a_n) \to 0 \quad \text{as} \quad n \to \infty.$$

Problem 4 (1pt + 1pt). Prove that the following series are convergent and evaluate them

$$(a) \quad \sum_{k=1}^{\infty} \frac{k^2}{2^k}; \qquad (b) \quad \sum_{k=1}^{\infty} \frac{1}{k2^k}.$$

Problem 5 (4pts; A-4, 1999). Evaluate the series

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{m^2 n}{3^m(n3^m + m3^n)}.$$

Problem 6 (2pts). Evaluate the series

$$\sum_{k=1}^{\infty} \frac{k^3 + 6k^2 + 11k + 5}{(k+3)!}.$$

Problem 7 (2pts). Let F_n be a Fibonacci sequence, that is,

$$F_1 = 1, \quad F_2 = 1, \quad F_n = F_{n-1} + F_{n-2}, \quad n = 3, 4, \ldots.$$

Evaluate the series

$$\sum_{n=2}^{\infty} \frac{F_n}{F_{n+1} F_{n-1}}.$$

Problem 8 (1pts). Let $\{a_n\}$ be a sequence of positive real numbers. Prove that if the series

$$\sum_{n=1}^{\infty} a_n$$

converges then so does the series

$$\sum_{n=1}^{\infty} \frac{\sqrt{a_n}}{n}.$$

Problem 9 (1pts). Let $\{a_n\}$ be a sequence of real numbers defined by $a_1 = 1$ and

$$a_{n+1} = \frac{a_n}{1 + n a_n}, \quad n = 2, 3, \ldots.$$

Prove that the series

$$\sum_{n=1}^{\infty} a_n$$

is convergent.