Math 3974 Problem Seminar Homework 4

Due Monday, March 11th

Problem 1 (1pt). The sequence \( \{a_n\} \) of real numbers is given by \( a_1 = 1 \), \( a_2 = \frac{3}{2} \), and

\[
na_n - a_{n-1} - (n-1)a_{n-2} = \frac{2n-3}{(n-1)(n-2)}, \quad n = 3, 4, \ldots.
\]

Find \( \lim_{n \to \infty} a_n \).

Problem 2 (2pts). Prove that

\[
\prod_{k=2}^{\infty} \left( 1 - \frac{(-1)^n}{n} \right) = \frac{1}{2}.
\]

Problem 3 (2pts; a half of an A-6 Putnam question). The sequence \( \{a_n\} \) of real numbers is defined by \( a_1 = 1 \), \( a_2 = 2 \), \( a_3 = 24 \), and for \( n \geq 4 \),

\[
a_n = \frac{6a_{n-1}^2a_{n-2} - 8a_{n-1}a_{n-2}^2}{a_{n-2}a_{n-3}}.
\]

Find an explicit formula for \( a_n \).

Problem 4 (1pt). Let \( a_n \) be a sequence of positive numbers such that

\[
a_n^2 - a_n - 2 \to 0.
\]

Prove that \( \lim_{n \to \infty} a_n = 2 \).

Problem 5 (1pt + 3pts). Compute the following limits

\[
(a) \lim_{n \to \infty} \sin^2 \left( \pi \sqrt{n^2 + 2019n} \right); \quad (b) \lim_{n \to \infty} \sin \left( 2\pi e^n! \right).
\]

Problem 6 (1pt + 1pt). Compute the following limits

\[
(a) \lim_{n \to \infty} \frac{1}{n^2} \sum_{k=1}^{n} k^2; \quad (b) \lim_{n \to \infty} \frac{n}{3^n} \sum_{k=1}^{n} \frac{3^{k-1}}{k}.
\]
Problem 7 (3pts). Let $a_n$ be a sequence of real numbers defined by

$$a_{n+1} = a_n(1 - a_n), \quad n = 1, 2, \ldots$$

and $a_1$ is such that $0 < a_1 < 1$. Prove that $\lim_{n \to \infty} na_n = 1$.

Problem 8 (2pts). Let $s_n$ be a sequence given by

$$s_n = 1 + \frac{1}{2} + \ldots + \frac{1}{n}, \quad n = 1, 2, \ldots$$

Prove that $\lim_{n \to \infty} (\ln s_n - \ln(\ln n)) = 0$.

Problem 9 (2pts). Let $F_n$ be a Fibonacci sequence, that is,

$$F_1 = 1, \quad F_2 = 1, \quad F_n = F_{n-1} + F_{n-2}, \quad n = 3, 4, \ldots$$

Compute $\lim_{n \to \infty} \sqrt[n]{F_n}$.

Problem 10 (6pts+2pts; B-6, 2006). Let $k$ be an integer number greater than 1. Suppose $a_0 > 0$, and define

$$a_{n+1} = a_n + \frac{1}{\sqrt[k]{a_n}}$$

for $n > 0$.

(a) Prove that

$$\lim_{n \to \infty} \left( a_{n+1}^{(k+1)/k} - a_n^{(k+1)/k} \right) = \frac{k + 1}{k}.$$

(b) Then, find the limit

$$\lim_{n \to \infty} \frac{a_n^{k+1}}{n^k}.$$