Math 3974 Problem Seminar Homework 4

Due Monday, March 11th

Problem 1 (1pt). The sequence $\{a_n\}$ of real numbers is given by $a_1 = 1$, $a_2 = \frac{3}{2}$, and

$$na_n - a_{n-1} - (n-1)a_{n-2} = \frac{2n-3}{(n-1)(n-2)}, \quad n = 3, 4, \ldots.$$

Find $\lim_{n\to\infty} a_n$.

Problem 2 (2pts). Prove that

$$\prod_{k=2}^{\infty} \left(1 - \frac{(-1)^n}{n}\right) = \frac{1}{2}.$$

Problem 3 (2pts; a half of an A-6 Putnam question). The sequence $\{a_n\}$ of real numbers is defined by $a_1 = 1$, $a_2 = 2$, $a_3 = 24$, and for $n \geq 4$,

$$a_n = \frac{6a_{n-1}^2 a_{n-2} - 8a_{n-1}a_{n-2}^2}{a_{n-2}a_{n-3}}.$$

Find an explicit formula for a_n.

Problem 4 (1pt). Let a_n be a sequence of positive numbers such that

$$a_n^2 - a_n - 2 \to 0.$$

Prove that $\lim_{n\to\infty} a_n = 2$.

Problem 5 (1pt + 3pts). Compute the following limits

$$(a) \lim_{n\to\infty} \sin^2\left(\pi\sqrt{n^2 + 2019n}\right); \qquad (b) \lim_{n\to\infty} \sin\left(2\pi e n!\right).$$

Problem 6 (1pt + 1pt). Compute the following limits

$$(a) \lim_{n\to\infty} \frac{1}{n2^n} \sum_{k=1}^{n} k2^k; \qquad (b) \lim_{n\to\infty} \frac{n}{3^n} \sum_{k=1}^{n} \frac{3^{k-1}}{k}.$$

Problem 7 (3pts). Let a_n be a sequence of real numbers defined by

$$a_{n+1} = a_n(1 - a_n), \quad n = 1, 2, \ldots$$

and a_1 is such that $0 < a_1 < 1$. Prove that $\lim_{n\to\infty} n a_n = 1$.

Problem 8 (2pts). Let s_n be a sequence given by

$$s_n = 1 + \frac{1}{2} + \ldots \frac{1}{n}, \quad n = 1, 2, \ldots.$$

Prove that $\lim_{n\to\infty} (\ln s_n - \ln(\ln n)) = 0$.

Problem 9 (2pts). Let F_n be a Fibonacci sequence, that is,

$$F_1 = 1, \quad F_2 = 1, \quad F_n = F_{n-1} + F_{n-2}, \quad n = 3, 4, \ldots.$$

Compute $\lim_{n\to\infty} \sqrt[n]{F_n}$.

Problem 10 (6pts+2pts; B-6, 2006). Let k be an integer number greater than 1. Suppose $a_0 > 0$, and define

$$a_{n+1} = a_n + \frac{1}{\sqrt[k]{a_n}}$$

for $n > 0$.

(a) Prove that

$$\lim_{n\to\infty} \left(a_{n+1}^{(k+1)/k} - a_n^{(k+1)/k} \right) = \frac{k+1}{k}.$$

(b) Then, find the limit

$$\lim_{n\to\infty} \frac{a_n^{k+1}}{n^k}.$$