Math 3974 Problem Seminar Homework 3

Due February 25, 2019

Problem 3.1. (Difficulty:1) If $x, y, z, w > 0$, then prove
$$(x^3y + y^3z + z^3w + w^3x)(x^2y^2 + z^2w^2) \geq 8x^2y^2z^2w^2.$$

Problem 3.2. (Difficulty:1) If $a, b, c > 0$, prove that
$$a^7 + b^7 + c^7 \geq a^4b^2c + b^4c^2a + c^4a^2b.$$

Problem 3.3. (Difficulty:2) If $a, b, c > 0$, prove that
$$a^6b + b^6c + c^6a \geq a^4b^2c + b^4c^2a + c^4a^2b.$$

Problem 3.4. (Difficulty:2) If $a, b, c, d > 0$ are given so that $abcd = 1$, prove that
$$a + b + c + d \leq a^2 + b^2 + c^2 + d^2.$$

Problem 3.5. (Difficulty:3) If $x, y, z > 0$, and $x + y + z = 1$, find the minimal value of
$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z}.$$

Problem 3.6. (Difficulty:2) For $a, b, c \geq 0$, prove that
$$\sqrt[3]{9(a+b+c)} \geq \sqrt[3]{a} + \sqrt[3]{b} + \sqrt[3]{c}.$$

Problem 3.7. (Difficulty:3) If a, b, c are the sides of a triangle, prove that
$$\frac{a}{b+c-a} + \frac{b}{c+a-b} + \frac{c}{a+b-c} \geq 3.$$

Problem 3.8. (Difficulty:3) Find the positive solutions of the system of equations
$$x_1 + \frac{1}{x_2} = 4, \quad x_2 + \frac{1}{x_3} = 1, \quad \cdots, \quad x_{99} + \frac{1}{x_{100}} = 4, \quad x_{100} + \frac{1}{x_1} = 1.$$

Problem 3.9. (Difficulty:3) Find the maximal value of $f(x) = \sin^4 x + \cos^4 x$ for $x \in \mathbb{R}$.

Problem 3.10. (Difficulty:2) Which is larger, $\log_2 3$ or $\log_3 5$?

Problem 3.11. (Difficulty:4) Prove that, for $n = 2, 3, 4, \ldots$
$$n! < \left(\frac{n+1}{2}\right)^n.$$