Problem Seminar, Fall 2018-Calculus II

1. (Difficulty 1) Let $f : [0, 1] \to \mathbb{R}$ be a continuous function. Prove that
$$\left(\int_0^1 f(x)dx\right)^2 \leq \int_0^1 f^2(x)dx.$$

2. (Difficulty 1) Find
$$\max_{f:[0,3]\to\mathbb{R} \text{ continuous and positive}} \frac{(\int_0^3 f(x))^3 cdx}{\int_0^3 f(x)^3 dx}.$$

3. (Difficulty 1) Let $f : \mathbb{R} \to (0, +\infty)$ be a continuous function. Show that
$$\int_0^1 f(x)dx \geq 2^{\int_0^1 \log_2 f(t)dt}.$$

4. (Difficulty 2) Let $f, g : \mathbb{R} \to \mathbb{R}$ such that f is bounded and not identically equal to zero. Show that if
$$f(x + y) + f(x - y) = 2f(x)g(y)$$
for all $x, y \in \mathbb{R}$ then $|g(y)| \leq 1$ for all $y \in \mathbb{R}$.

5. (Difficulty 2) Find all functions $f : \mathbb{R} \to \mathbb{R}$ which satisfy
$$f(x + y) + f(y + z) + f(z + x) \geq 3f(x + 2y + 3z)$$
for all $x, y, z \in \mathbb{R}$. *Hint: Using appropriate values for x, y, z compare $f(x)$ with $f(0)$.*

6. (Difficulty 3) Let $0 < a < b$. Prove that
$$\left|\int_a^b \sin x^2 \, \mathrm{d}x\right| < \frac{1}{a} + \frac{1}{b}.$$

7. (Difficulty 4) Prove that for every $r > 0$ the set
$$V_{f,g} = \{(x, y, z) \in \mathbb{R}^3 : (x - f(z))^2 + (y - g(z))^2 \leq r, \; z \in [a, b]\}$$
has the same volume for all continuous functions $f, g : [a, b] \to \mathbb{R}$.

8. (Difficulty 4) Let $f : \mathbb{R} \to (0, +\infty)$ be a continuous 1-periodic function. Prove that
$$\int_0^1 \frac{f(x)}{f(x + 1/2)} dx \geq 1.$$
Hint: Split the integral as $\int_0^1 = \int_0^{1/2} + \int_{1/2}^1$ and then change variables in the second integral so you would have same limits.

9. (Difficulty 5) Let $f : \mathbb{N} \to \mathbb{N}$ be a strictly increasing function such that $f(f(n)) = 3n$ for all n. Find $f(100)$.
Hint: First show that $f(1) = 2$ by eliminating the cases $f(1) = 1$ and $f(1) = m, m \geq 3$.

10. (Difficulty 5) Let $f : [0, 1] \to \mathbb{R}$ be a continuous function such that $\int_0^1 f(x)dx = 1$ and
$$\int_0^1 (1 - f(x))e^{-f(x)} dx \leq 0.$$
Show that $f(x) = 1$ for all $x \in [0, 1]$.