Problem Seminar, Fall 2018-Calculus I

1. (Difficulty 1) Evaluate the integral $\int_0^\pi \frac{\sin 2018x}{\sin x} dx$.

2. (Difficulty 1) Let $P(x)$ be a polynomial with real coefficients. Prove that
$$\int_0^\infty e^{-x} P(x) dx = P(0) + P'(0) + P''(0) + \dots.$$

3. (Difficulty 2) Evaluate the integral
$$\int_0^1 \int_0^1 \frac{1}{1+\max(x,y)^2} dx dy.$$

4. (Difficulty 2) Let $f : [0, +\infty) \to \mathbb{R}$ be a uniformly continuous function. Prove that there exist $a, b \geq 0$ such that
$$|f(x)| \leq ax + b,$$
for all $x \in [0, +\infty)$.

5. (Difficulty 2) Let $x_n > 0, n \in \mathbb{N}$. Show that
$$\sum_{n=1}^\infty x_n < \infty \text{ if and only if } \sum_{n=1}^\infty \frac{x_n}{x_1 + x_2 + \cdots + x_n} < \infty.$$

6. (Difficulty 3) Let S be a finite set. Consider the function
$$f(x) = \sum_{n \in S} \frac{\sin(nx)}{n}.$$
Prove that if $|f(x)| \leq 2018|x|$ for all $x \in \mathbb{R}$. Then S can have at most 2018 elements.

7. (Difficulty 3) [Putnam 2015 B1] Let f be a three times differentiable function (defined on $\mathbb{R}$ and real-valued) such that f has at least five distinct real zeros. Prove that $f + 6f' + 12f'' + 8f'''$ has at least two distinct real zeros.

   Hint: Consider the function $g(x) = 8e^{x/2} f(x)$.

8. (Difficulty 3) Let $f : [0, +\infty) \to \mathbb{R}$ be a continuous function such that $\lim_{x \to +\infty} x^2 f(x) = 1$. Show that
$$\lim_{n \to +\infty} \int_0^1 f(nx)\,\mathrm{d}x = 0$$

9. (Difficulty 4) Let $\mathcal{F}$ be a finite collection of open discs in $\mathbb{R}^2$ whose union contains a set $E \subset \mathbb{R}^2$. Show that there exists a pairwise disjoint subcollection $\{D_i = B(x_i, r_i)\}_{i=1}^n$ in $\mathcal{F}$ such that
$$E \subset \cup_{i=1}^n B(x_i, 3r_i).$$

10. (Difficulty 5)[Putnam 2000 A4] Show that the integral
$$\lim_{B \to \infty} \int_0^B \sin(x)\sin(x^2) dx$$
converges.

    Hint: Use integration by parts, $\sin(x)\sin(x^2) = -\frac{\sin x}{2x}(\cos x^2)'$.