Putnam preparation: Generating Functions
October 2nd, 2018 @UConn

Idea. Starting from a sequence $(a_n)_{n\geq 0}$, form the power series

$$\sum_{n\geq 0} a_n x^n$$

Find some way to compute or understand this power series. Profit!

Example 1. Use derivation/integration and geometric sums to compute generating power series for

(1) $a_n = 1$.

(2) $a_n = a^n$ for some fixed $a > 0$.

(3) $a_n = n$.

(4) $a_n = \begin{cases} \frac{1}{n} & \text{, if } n > 0 \\ C & \text{, if } n = 0 \end{cases}$

(5) $a_n = \frac{n}{n+1}$.

(6) $a_n = na^n$.

Example 2. Use Taylor series and generating functions to compute $1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2}$.
Hint: What does the series $(1 + 2x + 3x^2 + \ldots) \cdot \frac{1}{1-x}$ generate?

Example 3. Use multiplication of generating series to compute the number of ways of combining n total pieces of fruit to make salad if we have the following restrictions:

- The number of cherries must be even.
- The number of grapes must be a multiple of 5.
- There can be at most 3 apples.
- There can be at most 2 bananas.

Problem 4. What does Goldbach's conjecture say about

$$\Big(\sum_{p \text{ prime}} x^p\Big)^2?$$

Example 5 (#polynomialsarepowerseriestoo)**.**

$$\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}.$$

Hint: The term on the right is the coefficient of x^n in $(x+1)^{2n}$. Can you see the left hand side as the coefficient of x^n somewhere?

This method is particularly powerful for sequences defined by recurrences.

Example 6 (Generating Fibonacci)**.** Consider the Fibonacci sequence

$$f_{n+2} = f_{n+1} + f_n \quad \text{with } f_0 = 0 \quad \text{and} \quad f_1 = 1.$$

We can actually compute its generating power series. Compute

$$\left(\sum_{n \geq 0} f_n x^n \right) \cdot \left(1 - x - x^2\right).$$

[1]

Conclude that the formula for the generating power series is

$$\sum_{n \geq 0} f_n x^n = \frac{x}{1 - x - x^2}$$

We would like to actually compute the general term of this power series explicitly. The idea is to write it as a sum of two geometric functions, similar to what you would do if I asked you to integrate this function.

Solve $1 - x - x^2 = 0$. Its solutions are $x_{1,2} = ...$

Write

$$\frac{x}{1 - x - x^2} = \frac{A}{x - x_1} + \frac{B}{x - x_2} \tag{6.1}$$

for some real numbers A, B that are found by clearing denominators and identifying the coefficients of the resulting polynomials.

In our case, the equation is $-x = A \cdot (x - x_2) + B \cdot (x - x_1)$ leading to

$$\begin{cases} -1 = A + B \\ 0 = Ax_2 + Bx_1 \end{cases}$$

The solutions are $A =$ .... and $B =$ ....

Work backwards now to write $\frac{A}{x - x_1}$ and $\frac{B}{x - x_2}$ as sums of geometric series. Conclude that

$$\boxed{f_n = \frac{-A}{x_1} \cdot \frac{1}{x_1^n} + \frac{-B}{x_2} \frac{1}{x_2^n}}$$

[1] Recall from last time the **characteristic equation** of the Fibonacci sequence

$$x^2 - x - 1$$

It is not exactly what we work with here, but if you replace $x \mapsto \frac{1}{x}$, then they do look more alike. This is not a coincidence.

Remark 7. The same idea works for all "linear" recurrences with constant coefficients

$$a_{n+k} = c_{k-1}a_{n+k-1} + \ldots + c_0 a_n.$$

The generating power series will be of form

$$\sum_n a_n x^n = \frac{P(x)}{1 - c_{k-1}x - \ldots - c_0 x^k}$$

where P is a polynomial of degree at most $k-1$.

You factor the polynomial in the denominator. Say its roots are $x_1, \ldots, x_k$. They may repeat! Write $\frac{P(x)}{1-c_{k-1}x-\ldots-c_0x^k}$ as sum of fractions of form $\frac{A}{(x-x_i)^{r_i}}$, where r_i is at most the number of repetitions of x_i.

For example, if $k = 2$ and you factor $(x - x_1)(x - x_2)$ with $x_1 \neq x_2$ as in the Fibonacci case, then you only get fractions $\frac{A}{x-x_1}$ and $\frac{B}{x-x_2}$ as in (6.1).

But if $x_1 = x_2$, you may get fractions $\frac{A}{x-x_1}$ and $\frac{B}{(x-x_1)^2}$.

The final answer will look like $x_n = \bar{A}\frac{1}{x_1}^n + \bar{B}\frac{1}{x_2}^n$ in the first case, and $x_n = \bar{A}\frac{1}{x_1}^n + n\bar{B}\frac{1}{x_1}^n$ in the second case. $\square$

Example 8. Compute

$$\sum_{n\geq 0} x^n \sum_{k=0}^{n} \binom{k}{n-k}.$$

By convention $\binom{a}{b} = 0$ when $b < 0$ or $b > a$.

Hint: Switch the sums, replace $n \mapsto n - k$, and notice a product of two series. Reduce to a familiar sequence.

Problem 9 (Checkers jumping problem)**.** The xy plane is tiled with 1×1 squares – an infinite grid. On and below the x axis, we have checkers, one in every square. At every step, you can take one checker from a square, and jump over a horizontally or vertically adjacent checker onto an empty square. The checker that is jumped over is removed. Prove that you may never place a checker above the fourth line ($y = 4$) after finitely many moves.

Solution. We will show that we cannot reach the point $P = (0, 5)$. If we show this, then we are done. This is because translating the infinite board to the left or right does not change it.

Focusing on P, it is convenient to change coordinates so that it is now the origin. Now the checkers are all on and below the line $y = -5$.

Assign each square (i, j) in the plane a symbol $x^{|i|+|j|}$. When a checker is on it, we say that it has that symbol. This measures the horizontal+vertical distance from $P = (0, 0)$. We aim to get to P, so closer and closer to P, so it makes sense to measure this distance somehow.

At every step, we lose an x^n and x^{n+1} and gain an x^{n+2} or x^{n-1} depending on whether we jumped away from P or towards P respectively. There is also the possibility of gaining back x^n. This happens precisely when the piece that was jumped over was on the y-axis, so the new square is precisely as far from P as the old square.

We consider the invariant which is the sum of all the symbols on existing checkers. Let's call it **potential**. To get to P, at some point the potential has to be at least $1 = x^{|0|+|0|}$. Since we're aiming for a contradiction, the plan is as follows:

- Assign x a value such that the starting potential is < 1.
- Make sure that the moves never increase the potential.

Jumping over the y-axis, subtracts some x^{n+1} from the potential, so if $x > 0$, then we're good with this move. The most interesting/useful move is one that gets a checker closer to P. This subtracts $x^{n+1} + x^n - x^{n-1}$ from the potential. We ask that this value be 0, so that the potential stays the same (in particular it doesn't go up). The condition is $x^2 + x - 1 = 0$. The positive solution is $x = \frac{-1+\sqrt{5}}{2}$. For this x, the effect of jumping away from P is subtracting $x^n + x^{n+1} - x^{n+2} = x^n(1 + x - x^2) = x^n(1 + x - (1 - x)) = 2x^{n+1} > 0$, so the potential goes down.

Since none of the moves increases potential, let's check the starting potential. The total sum at the start is

$$x^5 + 3x^6 + 5x^7 + \ldots = x^5 \sum_{n \geq 0} (2n+1)x^n = x^5 \left(\frac{2}{(1-x)^2} - \frac{1}{1-x} \right)$$

$$= x^5 \cdot \frac{1+x}{(1-x)^2} = \frac{x^4}{(1-x)^2} = \left(\frac{x^2}{1-x} \right)^2 = 1.$$

Hmmm. Seems bad, right? Is it? The problem wants us to show that P cannot be reached in **finitely many** moves. Finitely many moves involve finitely many checkers. Therefore we can remove the (infinitely many) unused checkers from the board at the start, and the potential goes down a positive amount. $\square$