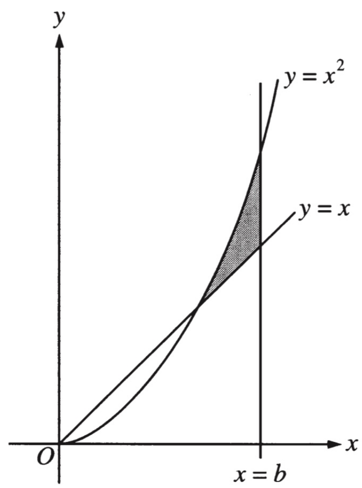
GRE PROBLEM SESSION IN CALCULUS/ANALYSIS (FALL 2018)

1. If $b > 0$ and $\int_0^b x dx = \int_0^b x^2 dx$ then the area of the shaded region is
   (a) 1/12
   (b) 1/6
   (c) 1/4
   (d) 1/3
   (e) 1/2



Solution: The answer is **(b)**. Notice that

$$0 = \int_0^b (x^2 - x)dx = \int_0^1 (x^2 - x)dx + \int_1^b (x^2 - x)dx.$$

So

$$\text{Shaded Area} = \int_1^b (x^2 - x)dx = \int_0^1 (x - x^2)dx = \frac{x^2}{2}\Big|_0^1 - \frac{x^3}{3}\Big|_0^1 = 1/2 - 1/3 = 1/6.$$

2. $\int_0^{\frac{\pi}{2}} \frac{1}{1+\tan x} dx =?$

(a) $\pi/2$

(b) $\pi/3$

(c) $\pi/4$

(d) $\pi^2/2$

(e) $\pi^2/4$

Solution: The answer is **(c)**. Let

$$I = \int_0^{\frac{\pi}{2}} \frac{1}{1+\tan x} dx.$$

The trick here is to rewrite

$$I = \int_0^{\frac{\pi}{2}} \frac{1}{1+\frac{\sin x}{\cos x}} dx = \int_0^{\frac{\pi}{2}} \frac{1}{\frac{\cos x}{\cos x}+\frac{\sin x}{\cos x}} dx = \int_0^{\frac{\pi}{2}} \frac{\cos x}{\cos x+\sin x}.$$

Then note that if

$$J = \int_0^{\frac{\pi}{2}} \frac{\sin x}{\cos x+\sin x} dx,$$

we have that

$$J - I = \int_0^{\frac{\pi}{2}} \frac{\sin x - \cos x}{\cos x+\sin x} dx = \int_0^{\frac{\pi}{2}} \frac{(\sin x+\cos x)'}{\cos x+\sin x} dx = \log(\sin x+\cos x)\Big|_0^{\pi/2} = 0.$$

So

$$J = I. \tag{1}$$

On the other hand

$$J + I = \int_0^{\frac{\pi}{2}} \frac{\sin x+\cos x}{\cos x+\sin x} dx = \pi/2. \tag{2}$$

By (1) and (2) we deduce that $I = \pi/4$

3. Let $p : \mathbb{R} \to \mathbb{R}$ be a polynomial such that $p(0) = p(2) = 3$ and $p'(0) = p'(2) = -1$. Then $\int_0^2 xp''(x)dx =$

   (a) -3

   (b) -2

   (c) -1

   (d) 1

   (e) 2

Solution: The answer is **(b)**. The idea here is to use integration by parts. Then

$$\int_0^2 xp''(x)dx = xp'(x)\Big|_0^2 - \int_0^2 p'(x)dx = 2p'(2) - (p(2) - p(0)) = 2(-1) = -2.$$

4. At how many points the graphs of the functions x^{200} and 5^x intersect?

   (a) 0
   (b) 1
   (c) 3
   (d) 2
   (e) 5

Solution: The answer is **(c)**. Drawing the graphs of these functions it is easy to convince yourselves that they intersect at two points $r_1 < 0 < r_2$ close to the origin. Observe though, that after on that after r_2, $x^{200} > 5^x$. Nevertheless 5^x grows faster than x^{200} as x goes to infinity hence there should be another root $r_3 > r_2$.

5. How many linearly independent solutions does the following equation has

$$e^{-x}y'' + x^7y' - \cos(x)y = 0?$$

(a) 0

(b) 1

(c) 2

(d) undecidable

(e) infinitely many

Solution: The answer is **(c)**. This is a nice application of the IVP theorem for linear differential equations of second order. Consider the following two IVPs

$$\begin{aligned} &e^{-x}y'' + x^7y' - \cos(x)y = 0 \\ &y(0) = 1, \\ &y'(0) = 0. \end{aligned}$$

and

$$\begin{aligned} &e^{-x}y'' + x^7y' - \cos(x)y = 0 \\ &y(0) = 0, \\ &y'(0) = 1. \end{aligned}$$

Since all the coefficient functions are continuous in $\mathbb{R}$, the IVP(1) has a solution y_1 and IVP(2) has a solution y_2 which are easily seen to be linearly independent.

6. Let $g(x) = e^{2x+1}, x \in \mathbb{R}$. Then $\lim_{x\to 0} \frac{g(g(x))-g(e)}{x} =$

(a) $2e$

(b) $4e^2$

(c) e^{2e+1}

(d) $4e^{2e+2}$

(e) 2^{2e+1}

Solution: The answer is **(d)**. The idea here is to notice that you can use L' Hopital's rule since $g(0) = e$. Hence

$$\lim_{x\to 0} \frac{g(g(x)) - g(e)}{x} = \lim_{x\to 0} g'(g(x))g'(x) = g'(g(0))g'(0) = 2e^{2e+1}2e = 4e^{2e+2}.$$

7. Decide which of the following hold:

(i) $\ln x \leq C\sqrt{x}$ for some constant C and $x \geq 1$,

(ii) $\sum_{k=1}^{n} k^2 \leq Cn^2$ for some constant C and $n \geq 1$,

(iii) $|\sin x - x| \leq C|x^3|$ for some constant C.

(a) None

(b) (i)+(iii)

(c) (iii)

(d) (i)

(e) all

Solution: The answer is **(a)**. For (i) its good to remember the logarithmic is smaller than any positive power for large x. In this case note that if $h(x) = \ln x - C\sqrt{x}$ then $h(1) = -C$ which is negative for all $C > 0$. Taking derivatives we see that $h'(x) = \frac{2-C\sqrt{x}}{2x}$. Since $2 - C\sqrt{x} \leq 0$ for $C \geq 2$ we can choose $C = 2$.

For (ii) it's useful to remember that $\sum_{k=1}^{n} k^2 \approx n^3$, to be more precise $\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$. Hence (ii) can't be true.

The statement (iii) is correct and it follows easily if you remember the Taylor expansion of $\sin x$, that is

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - ..$$

8. Let f, g real functions such that $g(x) = \int_0^x f(y)(y-x)\,dy$. If g is three times continuously differentiable how many times continuously differentiable is f?

(a) 1

(b) 2

(c) 3

(d) 4

(e) 5

Solution: The answer is **(a)**. For this problem it is useful to remember Feynman's trick:

$$\frac{d}{dx}\int_{a(x)}^{b(x)} f(x,t)dt = f(x,b(x))b'(x) - f(x,a(x))a'(x) + \int_{a(x)}^{b(x)} \frac{\partial}{\partial x} f(x,t)dt.$$

Using Feynman's trick, it follows easily that

$$g'(x) = -\int_0^x f(y)dy.$$

Hence $g''(x) = -f(x)$ and $g'''(x) = -f'(x)$. Since we know that g is three times continuously differentiable, this is where we stop. Hence f' is continuously differentiable.

9. Let $f : \mathbb{Z} \to \mathbb{R}$ such that $f(0) = 0$ and for all $x \in \mathbb{Z}$

$$f(x) = \frac{f(x-1) + f(x+1)}{2}. \tag{3}$$

Which of the following hold?

(i) The graph of f is contained in a line.

(ii) f is strictly increasing

(iii) f is constant.

(a) None

(b) (i) only

(c) (ii) only

(d) (i) and (ii)

(e) (i) and (iii)

Solution: The answer is **(b)**. First note that the function $f(x) = mx$ for $m \in \mathbb{Z}$ satisfies

$$\frac{f(x-1) + f(x+1)}{2} = \frac{m(x-1) + m(x+1)}{2} = \frac{2mx}{2} = mx = f(x).$$

So f satisfies (3). Notice that functions of the form mx are not necessarily increasing or constant. So we can eliminate (c), (d) and (e). Therefore we only have to show that the functions of the form mx are the only possible solutions to equation (3). To see this consider the function $g(x) = f(x+1) - f(x)$. Then

$$\begin{aligned} g(x) = f(x+1) - f(x) &= \frac{f(x+2) + f(x)}{2} - f(x) = \frac{f(x+2) - f(x)}{2} \\ &= \frac{2f(x+2) - f(x+2) - f(x)}{2} = f(x+2) - \frac{f(x+2) + f(x)}{2} \\ &= f(x+2) - f(x+1) = g(x+1). \end{aligned}$$

So g is constant and $f(m) = mf(1)$ for $m \in \mathbb{Z}$.

10. Let $B \subset \mathbb{R}, B \neq \emptyset$, be bounded. If $\sup B \notin B$, which of the following is true?

(a) B is closed

(b) B is not open

(c) There is a sequence in B converging to $\sup B$

(d) No sequence in B converges to $\sup B$

(e) There is an open interval containing $\sup B$ but not containing any point of B.

Solution: The answer is **(c)**. The set B cannot be closed because closed bounded sets contain their supremum. It doesn't have to be open necessarily, for example B can be $[0, 1)$. Statement (d) is false in general, and (e) is not necessarily true, for example B can be $[0, 1)$.

11. $\int_0^\pi \frac{\sin 100x}{\sin x} dx =?$

(a) 100

(b) 1

(c) 1/2

(d) 0

(e) $\sqrt{3}/2$

Solution: The answer is **(d)**. To see this first do a change of variables $y = x - \pi/2$:

$$\int_{-\pi/2}^{\pi/2} \frac{\sin 100(y + \pi/2)}{\sin(y + \pi/2)} dy = \int_{-\pi/2}^{\pi/2} \frac{\sin(100y + 50\pi)}{\cos(y)} dy = \int_{-\pi/2}^{\pi/2} \frac{\sin(100y)}{\cos(y)} dy.$$

Now notice that the function $\frac{\sin(100y)}{\cos(y)}$ is odd.