MATH 3974 PROBLEM SEMINAR HOMEWORK 6, DUE NOVEMBER 13

1(1 pt). Prove that

$$1-\frac{1}{x}\leq \ln x\leq x-1,$$

for all $x\geq 1$.

2 (1 pt). Prove that

$$e^x\geq 1+x^2,$$

for all $x\geq 0$.

3 (2 pts). Let a,b,c be the side lengths of a triangle with the property that for any positive integer n, the numbers a^n, b^n, c^n can also be the side lengths of a triangle. Prove that the triangle is necessarily isosceles.

4 (3 pts). Prove that the positive real numbers a,b,c are the side lengths of a triangle if and only if

$$a^2+b^2+c^2<2\sqrt{a^2b^2+b^2c^2+c^2a^2}.$$

5 (2 pts). Prove that if $z\in\mathbb{C}$ satisfies $\mathrm{Re} z<\frac{1}{2}$, then

$$\left|\frac{z}{1-z}\right|<1.$$

6 (3 pts). Prove that

$$\left|\frac{z-w}{1-\bar{z}w}\right|<1$$

for any complex numbers z and w so that $|z|<1$ and $|w|<1$.

7 (3 pts). Let $z_1,..,z_n\in\mathbb{C}$ be so that $|z_1|=|z_2|=...=|z_n|=1$. Prove that the number

$$\omega=(z_1+z_2+...+z_n)\left(\frac{1}{z_1}+\frac{1}{z_2}+....+\frac{1}{z_n}\right)$$

is real and $0\leq\omega\leq n^2$.

8 (3 pts). Let $f:[0,1]\to\mathbb{R}$ be a decreasing function. Prove that for any $a\in(0,1)$

$$a\int_0^1 f(x)\,dx\leq\int_0^a f(x)\,dx.$$

9 (4 pts). Let $(x_n)_{n\geq 1}$ be a sequence of real numbers satisfying

$$x_{n+m} \leq x_n + x_m \qquad \text{for all } n, m \geq 1.$$

Prove that

$$\lim_{n\to\infty} \frac{x_n}{n} \quad \text{exists and is finite.}$$

10 (4 pts). Which of

$$\sqrt{2+\sqrt{3+\sqrt{2+\ldots}}} \text{ or } \sqrt{3+\sqrt{2+\sqrt{3+\ldots}}}$$

is larger? Each number contains n succesive square roots.