Math 3974 Problem Seminar Homework 4

Due October 16, 2018

Problem 4.1. (Difficulty: $1+2+1$) Find the ordinary power series $\sum_{n\geq 0} a_n x^n$ generating functions of each of the following sequence, in simple, closed form. In each case the sequence is defined for all $n \geq 0$.

(a) $a_n = n^2$.
(b) $a_n = P(n)$, where P is a given polynomial of degree m.
(c) $a_n = 5 \cdot u^n - 3 \cdot 4^n$.

Problem 4.2. (Difficulty: $1+1+1+1$) If $f(x) = \sum_{n\geq 0} a_n x^n$ is the ordinary power series generating function of the sequence $\{a_n\}_{n\geq 0}$, then express simply, in terms of $f(x)$, the ordinary power series generating functions of the following sequence (start with $n = 0, 1, 2, \dots$)

(a) $\{na + c\}$.
(b) $\{0, 0, 1, a_3, a_4, a_5, \dots\}$.
(c) $\{a_0, 0, a_2, 0, a_4, \dots\}$.
(d) $\{a_{n+2} - a_{n+1} - a_n\}$.

Problem 4.3. (Difficulty: 1) Find the x^n-coefficient of $\dfrac{1}{(1-ax)(1-bx)}$ where $a \neq b$.

Problem 4.4. (Difficulty: 2) Find the ordinary power series generating function of the sequence

$$a_0 = 0,\ a_1 = 1,\ a_{n+2} = 3a_{n+1} - 2a_n \text{ for } n \geq 0.$$

Problem 4.5. (Difficulty: 3) Let $f(n)$ be the number of subsets of $\{1, \dots, n\}$ that contain no two consecutive elements, for positive integer n. Find the recurrence that is satisfied by these numbers, and then give a closed formula for these numbers $f(n)$.

Problem 4.6. (Difficulty: 4) Let $x^{(n)} = x(x-1)\cdots(x-n+1)$ for n a positive integer, and let $x^{(0)} = 1$. Prove that

$$(x+y)^{(n)} = \sum_{k=0}^{n} \binom{n}{k} x^{(k)} y^{(n-k)}.$$

Problem 4.7. (Difficulty: 3) A function f is defined for all $n \geq 1$ by the relation

(1) $f(1) = 1$.
(2) $f(2n) = f(n)$.
(3) $f(2n+1) = f(n) + f(n+1)$.

Let

$$F(x) = \sum_{n\geq 1} x^{n-1}$$

be the generating function of the sequence. Show that

$$F(x) = (1 + x + x^2)F(x^2)$$

and conclude that

$$F(x) = \prod_{j\geq 0} \left(1 + x^{2^j} + x^{2^{j+1}}\right).$$

Problem 4.8. (Difficulty: 7) Sum the series

$$\sum_{m\geq 1} \sum_{n\geq 1} \frac{3^{-m} m^2 n}{3^m n + 3^n m}.$$