

Math 3974 Problem Seminar Homework 1

Due September 11, 2018

Problem 1.1. (Difficulty:1) Compute the sum

$$\frac{1}{1 \cdot 2 \cdot 3} + \frac{1}{2 \cdot 3 \cdot 4} + \cdots + \frac{1}{98 \cdot 99 \cdot 100}.$$

Problem 1.2. (Difficulty:1) Compute the sum

$$1 \cdot 2 \cdot 3 + 2 \cdot 3 \cdot 4 + \cdots + 98 \cdot 99 \cdot 100.$$

Problem 1.3. (Difficulty:1) Compute the sum

$$\sum_{n=1}^N n \cdot n!$$

Problem 1.4. (Difficulty:2) Compute the sum

$$\frac{1}{\sqrt{2} + 1} + \frac{1}{\sqrt{3} + \sqrt{2}} + \cdots + \frac{1}{\sqrt{99} + \sqrt{100}}.$$

Problem 1.5. (Difficulty:2) Find the product

$$\prod_{n=2}^{2016} \left(1 - \frac{1}{n^2}\right).$$

Problem 1.6. (Difficulty:3) Evaluate

$$\sum_{n=1}^{2016} \frac{1}{\sqrt[3]{n^2} + \sqrt[3]{n(n+1)} + \sqrt[3]{(n+1)^2}}.$$

Problem 1.7. (Difficulty:3) Given that a_1, a_2, \dots, a_n are positive numbers in arithmetic progression, show that

$$\frac{1}{a_1 a_2} + \frac{1}{a_2 a_3} + \cdots + \frac{1}{a_{n-1} a_n} = \frac{n-1}{a_1 a_n}$$

and

$$\frac{1}{\sqrt{a_1} + \sqrt{a_2}} + \cdots + \frac{1}{\sqrt{a_{n-1}} + \sqrt{a_n}} = \frac{n-1}{\sqrt{a_1} + \sqrt{a_n}}.$$

Problem 1.8. (Difficulty:3) Evaluate

$$\sum_{k=0}^r (-1)^k \binom{n}{k}$$

for $0 \leq r \leq n$.

Problem 1.9. (Difficulty:3) Evaluate the infinite series

$$\sum_{n=0}^{\infty} \frac{n}{n^4 + n^2 + 1}.$$

Hint: $x^4 + x^2 + 1$ is not an irreducible polynomial. Remember this!

Problem 1.10. (Difficulty:3) (Putnam 1977B1) Evaluate the infinite product

$$\prod_{n=2}^{\infty} \frac{n^3 - 1}{n^3 + 1}.$$

Problem 1.11. (Difficulty:3) Find the value of the following infinite tower of exponents:

$$\sqrt{2}^{\sqrt{2}^{\sqrt{2}^{\sqrt{2}^{\dots}}}}$$

interpreted as the limit of the infinite sequence $1, \sqrt{2}, \sqrt{2}^{\sqrt{2}}, \sqrt{2}^{\sqrt{2}^{\sqrt{2}}}, \dots$. Be rigorous about your argument.

Problem 1.12. (Difficulty:4) If $x \neq 0$, prove that

$$\frac{\sin x}{x} = \prod_{n=1}^{\infty} \cos\left(\frac{x}{2^n}\right).$$

Problem 1.13. (Difficulty:4) Evaluate the sum

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{1^3 + 2^3 + 3^3 + \dots + n^3}}.$$

Hint: First compute the sum $1^3 + 2^3 + 3^3 + \dots + n^3$ for small n , did you see some pattern?

Problem 1.14. (Difficulty:4) Find the limit

$$\lim_{n \rightarrow \infty} \prod_{k=0}^n \left(1 + \frac{1}{3^{2^k}}\right).$$

Problem 1.15. (Difficulty:6) Determine whether the following value is finite

$$\sqrt{1^2 + \sqrt{2^2 + \sqrt{3^2 + \sqrt{\dots}}}}$$

Problem 1.16. (Difficulty:8) (Putnam 2015A3) Let $a_0 = \frac{5}{2}$ and $a_k = a_{k-1}^2 - 2$ for $k \geq 1$. Compute the following in a closed form

$$\prod_{k=0}^{\infty} \left(1 - \frac{1}{a_k}\right).$$

Hint: Try to get the expression of a_k first. Try out a few terms and you can see the pattern.