SOLUTIONS TO 2018 STUART SIDNEY CALCULUS COMPETITION

Tuesday 27 March, 2018, 6:30–8:00 p.m.

Please show enough of your work so your line of reasoning will be clear. Numerical answers will receive no credit if they are not adequately supported. Calculators are welcome, but unlikely to be very useful. Have fun, and good luck!

1. Find (any) polynomial $f(x)$ for which $x = -1$ and $x = 2$ are local minimum points and $x = -2$ and $x = 1$ are local maximum points. Justify your answer.

   Solution. The condition implies that $f'(x) = 0$ at $x = -1, -2, 1, 2$. Moreover, the condition suggests that the derivative $f'(x)$ is positive when $x > 2$, $x \in (-1, 1)$, and $x < -2$; and $f'(x)$ is negative when $x \in (1, 2)$ and $x \in (-2, -1)$. For example, we may take $f'(x) = (x+1)(x+2)(x-1)(x-2) = (x^2-1)(x^2-4) = x^4 - 5x^2 + 4$. Integrating, we have $f(x) = \frac{1}{5}x^5 - \frac{5}{3}x^3 + 4x$.

2. Consider the ellipse $x^2 + \frac{y^2}{4} = 1$. What is the area of the smallest diamond shape with two vertices on the x-axis and two vertices on the y-axis that contains this ellipse?

   Solution. Clearly, the four sides of the minimal diamond shape are tangent to the ellipse. To answer this question, we may rescale the y-axis by $\frac{1}{2}$ so that the question becomes inscribing a circle. It is then easy to guess that we should just take a square with side length 2. The area of the square is $2^2 = 4$. Scaling back, the area of the diamond shape is $4 \times 2 = 8$. Indeed we can prove that taking the square will achieve the min if the tangent point of the side of the diamond to the circle has argument θ, the area of the diamond that inscribes the circle is $4 \times \frac{1}{2}\frac{1}{\sin\theta}\frac{1}{\cos\theta} = \frac{4}{\sin 2\theta}$, which achieves minimum 4 when $\theta = 45°$.

3. Evaluate the definite integral

   $$I = \int_0^1 (\sqrt[20]{1-x^{18}} - \sqrt[18]{1-x^{20}})dx.$$

   Solution. Observe that if $f(x) = \sqrt[20]{1-x^{18}}$ for $x \in [0,1]$ then $f^{-1}(x) = \sqrt[18]{1-x^{20}}$ for $x = [0,1]$. The function $f(x)$ is monotonic decreasing. The integral $\int_0^1 f(x)dx$ computes the area that is below the graph of $f(x)$, whereas the integral $\int_{f(1)}^{f(0)} f^{-1}(x)dx$ computes the area that is to the left of the graph of $f(x)$. But the they compute the same area. So the total integral $I = 0$.

   Alternative Solution. Make a change of variable $y(x) = \sqrt[20]{1-x^{18}}$. Then $y(0) = 1$ and $y(1) = 0$. So

   $$\int_0^1 \sqrt[18]{1-x^{20}}dx = \int_1^0 yx'(y)dy = (yx(y))\Big|_1^0 - \int_1^0 x(y)dy = 0 - \int_1^0 x(y)dy = \int_0^1 \sqrt[20]{1-y^{18}}dy;$$

   so $I = 0$.

4. Evaluate the integral

   $$I = \int \frac{x\cos x - \sin x}{x^2 + \sin^2 x}dx.$$

You need to justify your answer.

Solution. Divide the denominator and numerator by x^2 to get

$$I = \int \frac{\frac{\cos x}{x} - \frac{\sin x}{x^2}}{1 + \left(\frac{\sin x}{x}\right)^2} dx.$$

This suggests to make the substitution $y = \frac{\sin x}{x}$. We check that

$$\frac{dy}{dx} = \frac{(\cos x)x - \sin x}{x^2} = \frac{\cos x}{x} - \frac{\sin x}{x^2},$$

which is exactly the numerator of the integral. Hence,

$$\begin{aligned} I &= \int \frac{dy}{1+y^2} \\ &= \arctan y + C \\ &= \arctan\left(\frac{\sin x}{x}\right) + C. \end{aligned}$$

5. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function and

$$g(x) = f(x) \int_0^x f(t)\,dt.$$

Prove that if g is a non-increasing function (meaning $x_1 \geq x_2$ implies $g(x_1) \leq g(x_2)$), then f is identically equal to zero.

Solution. By the fundamental theorem of calculus,

$$F(x) = \int_0^x f(t)\,dt$$

is differentiable, $F'(x) = f(x)$ and $F(0) = 0$. We can write

$$g(x) = F'(x)\,F(x) = \frac{1}{2}\left(F^2\right)'(x).$$

The fact that $(F^2)'$ is non-increasing means that $(F^2)'' \leq 0$, so F^2 is concave down. So for the differentiable function $F(x)$ with $F(0) = 0$, the function F^2 is concave down. As $F^2 \geq 0$ and $F^2(0) = 0$, we conclude that F^2 achieves its minimum value at $x = 0$. However, as F^2 is concave down, this can only happen if $F^2 = 0$, hence $f = 0$.

6. Prove that the function $f : \mathbb{R}^2 \to \mathbb{R}$ given by

$$f(x,y) = x^4 + 6x^2y^2 + y^4 - \frac{9}{4}x - \frac{7}{4}y$$

achieves its minimum value, and determine all the points $(x, y) \in \mathbb{R}^2$ at which it is achieved.

Solution. The function is continuous on $\mathbb{R}^2$ and

$$\lim_{(x,y)\to\infty} f(x,y) = \infty,$$

so f must achieve its minimum value at some point (x, y). At this point, $\frac{\partial f}{\partial x} = 0$ and $\frac{\partial f}{\partial y} = 0$, which gives

$$\begin{aligned} 4x^3 + 12xy^2 - \frac{9}{4} &= 0 \\ 12x^2y + 4y^3 - \frac{7}{4} &= 0. \end{aligned}$$

Divide both equations by 4 to get

$$\begin{aligned} x^3 + 3xy^2 &= \frac{9}{16} \\ y^3 + 3x^2y &= \frac{7}{16}. \end{aligned}$$

Observe that adding the two equations yields a perfect cube, by the formula

$$(x + y)^3 = x^3 + 3x^2y + 3xy^2 + y^3.$$

Hence, we obtain

$$(x + y)^3 = 1.$$

Similarly, by subtracting the two equations and using that

$$x^3 + 3xy^2 - 3x^2y - y^3 = (x - y)^3,$$

it follows that

$$(x - y)^3 = \frac{2}{16} = \frac{1}{8}.$$

So the system becomes

$$\begin{aligned} x + y &= 1 \\ x - y &= \frac{1}{2}, \end{aligned}$$

which has unique solution $(x, y) = \left(\frac{3}{4}, \frac{1}{4}\right)$.

7. Compute the integral $\iint_D x\,dx\,dy$, where

$$D = \left\{(x, y) \in \mathbb{R}^2 : x > 0,\ 1 \le xy \le 2,\ 1 \le \frac{y}{x} \le 2\right\}.$$

Solution. We make the substitution

$$\begin{aligned} u &= xy \\ v &= \frac{y}{x}. \end{aligned}$$

This turns the domain D into a rectangle R in (u, v) :

$$R = \left\{(u, v) \in \mathbb{R}^2 :\ 1 \le u, v \le 2\right\}.$$

The Jacobian of this transformation is

$$J = \det \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix}.$$

Solving for x, y we get from $u = xy$ and $v = y/x$ that

$$\begin{aligned} x &= \sqrt{\frac{u}{v}} \\ y &= \sqrt{uv}. \end{aligned}$$

Now we can compute the Jacobian

$$\begin{aligned} J &= \det \begin{pmatrix} \frac{1}{2\sqrt{uv}} & -\frac{1}{2v}\sqrt{\frac{u}{v}} \\ \frac{1}{2}\sqrt{\frac{v}{u}} & \frac{1}{2}\sqrt{\frac{u}{v}} \end{pmatrix} \\ &= \frac{1}{2v}. \end{aligned}$$

Using the change of variables formula it follows that

$$\begin{aligned} \iint_D x dx dy &= \iint_R \sqrt{\frac{u}{v}} \frac{1}{2v} du dv \\ &= \frac{1}{2} \left(\int_1^2 \sqrt{u} du \right) \left(\int_1^2 \frac{1}{v\sqrt{v}} dv \right) \\ &= \frac{1}{2} \left(\frac{2}{3}(2^{3/2} - 1) \right) \left(-2 \left(\frac{1}{\sqrt{2}} - 1 \right) \right) \\ &= \frac{5\sqrt{2} - 6}{3}. \end{aligned}$$

8. Let $s \in \mathbb{R}$. Prove that

$$\sum_{n \geq 1} \left(n^{1/n^s} - 1 \right)$$

converges if and only if $s > 1$.

Solution The sum can be rewritten as

$$\sum_{n \geq 1} (e^{\frac{\ln n}{n^s}} - 1).$$

By the limit comparison test, using that $\lim_{x \to 0} \dfrac{e^x - 1}{x} = 1$, it is enough to treat

$$\sum_{n \geq 1} \frac{\ln n}{n^s}.$$

If $s \leq 1$, then $\dfrac{\ln n}{n^s} > \dfrac{1}{n}$ for all large n.

If $s > 1$, then $\ln n < n^{\frac{s-1}{2}}$ for large n, and $\dfrac{\ln n}{n^s} < \dfrac{1}{n^{(s+1)/2}}$.

The series modeled on the latter term converges since $\dfrac{s+1}{2} > 1$.