2018 STUART SIDNEY CALCULUS COMPETITION

Tuesday 27 March, 2018, 6:30–8:00 p.m.

Please show enough of your work so your line of reasoning will be clear. Numerical answers will receive no credit if they are not adequately supported. Calculators are welcome, but unlikely to be very useful. Have fun, and good luck!

1. Find (any) polynomial $f(x)$ for which $x = -1$ and $x = 2$ are local minimum points and $x = -2$ and $x = 1$ are local maximum points. Justify your answer.

2. Consider the ellipse $x^2 + \frac{y^2}{4} = 1$. What is the area of the smallest diamond shape with two vertices on the x-axis and two vertices on the y-axis that contains this ellipse?

3. Evaluate the definite integral
$$I = \int_0^1 \left(\sqrt[20]{1 - x^{18}} - \sqrt[18]{1 - x^{20}} \right) dx.$$

4. Evaluate the integral
$$I = \int \frac{x \cos x - \sin x}{x^2 + \sin^2 x} dx.$$
You need to justify your answer.

5. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function and
$$g(x) = f(x) \int_0^x f(t)\, dt.$$
Prove that if g is a non-increasing function (meaning $x_1 \geq x_2$ implies $g(x_2) \geq g(x_1)$), then f is identically equal to zero.

6. Prove that the function $f : \mathbb{R}^2 \to \mathbb{R}$ given by
$$f(x, y) = x^4 + 6x^2y^2 + y^4 - \frac{9}{4}x - \frac{7}{4}y$$
achieves its minimum value, and determine all the points $(x, y) \in \mathbb{R}^2$ at which it is achieved.

7. Compute the integral $\iint_D x\, dxdy$, where
$$D = \left\{ (x, y) \in \mathbb{R}^2 : x > 0,\ 1 \leq xy \leq 2,\ 1 \leq \frac{y}{x} \leq 2 \right\}.$$

8. Let $s \in \mathbb{R}$. Prove that
$$\sum_{n \geq 1} \left(n^{1/n^s} - 1 \right)$$
converges if and only if $s > 1$.

Now wasn't that fun?