GRE prep questions in Analysis

1. Let \( f_n(x) = \frac{x^n}{1+x^n}, \ x \in [0, 1] \). Which of the following statements hold

(A) \( f_n \) converges pointwise to a function \( f : [0, 1] \to \mathbb{R} \).

(B) \( f_n \) converges uniformly to a function \( f : [0, 1] \to \mathbb{R} \).

(C) \[ \lim_{n \to \infty} \int_0^1 f_n(x) \, dx = \int_0^1 \lim_{n \to \infty} f_n(x) \, dx. \]
2. For which \( x \) does the series

\[ \sum_{n \in \mathbb{N}} \frac{n!x^{2n}}{n^n(1+x^{2n})} \]

converge?

(A) \( \{0\} \)
(B) \( \mathbb{R} \)
(C) \( (-1, 1) \)
(D) \( [-1, 1] \)
3. In the figure above we let \( r \) and \( s \) increase while keeping one sided fixed with length 1 and the obtuse angle fixed at 110 degrees. Then

\[
\lim_{r,s \to \infty} s - r
\]

(A) = 0,
(B) \( \in (0, 1) \),
(C) = 1,
(D) \( \in (1, \infty) \),
(E) = \( \infty \).
4. Let \( f : (-1, 4) \to \mathbb{R} \) be a continuously differentiable function such that \( f(3) = 5 \) and \( f'(x) \geq -1 \) for all \( x \in (1, 4) \). What is the greatest possible value of \( f(0) \)?

(A) 3
(B) 4
(C) 5
(D) 8
(E) 11
5. Which of the following equations has the greatest number of real solutions?

(A) $x^3 = 10 - x$
(B) $x^2 + 5x - 7 = x + 8$
(C) $7x + 5 = 1 - 3x$
(D) $e^x = x$
(E) $\sec x = e^{-x^2}$
6. Find the limit

\[ \lim_{z \to 0} \frac{z^2}{z^2}, \ z \in \mathbb{C}. \]

(A) 0
(B) 1
(C) \(i\)
(D) \(\infty\)
(E) The limit does not exist.
7. Let $S \subset \mathbb{R}$. Which of the following statements is necessarily true?

(A) For all $t, s \in S$ there exists a continuous function $f : [0, 1] \to S$ such that $f(0) = s$ and $f(1) = t$.

(B) For each $x \notin S$, there exists an open set $U \subset \mathbb{R}$ such that $u \in U$ and $U \cap S = \emptyset$.

(C) \{x \in S : \text{there exists an open set } V \text{ such that } x \in V \subset S \} \text{ is an open subset of } \mathbb{R}.

(D) \{x \notin S : \text{there exists an open set } W \text{ such that } x \in W \text{ and } W \cap S = \emptyset \} \text{ is a closed set.}

(E) $S$ is the intersection of all closed subsets of $\mathbb{R}$ that contain $S$. 
8. How many positive solutions does the equation \( \cos(97x) = x \) have?

(A) 1  
(B) 15  
(C) 31  
(D) 49  
(E) 0
9. Let $f, g$ real functions such that $g(x) = \int_0^x f(y)(y - x) \, dy$. If $g$ is three times continuously differentiable how many times continuously differentiable is $f$?

(A) 1  
(B) 2  
(C) 3  
(D) 4  
(E) 5
10. Let $f, g$ be twice differentiable functions on $\mathbb{R}$ such that $f'(x) > g'(x)$ for all $x > 0$. Which of the following does it hold for $x > 0$:

(A) $f(x) > g(x)$
(B) $f''(x) > g''(x)$
(C) $f(x) - f(0) > g(x) - g(0)$
(D) $f'(x) - f'(0) > g'(x) - g'(0)$
(E) $f''(x) - f''(0) > g''(x) - g''(0)$
11. How many continuous functions \( f : [-1, 1] \to \mathbb{R} \) do they exist such that \( f(x)^2 = x^2 \) for all \( x \in [-1, 1] \)?

(A) 1
(B) 2
(C) 3
(D) 4
(E) 5
12. Suppose that \( f \) is twice differentiable on \( \mathbb{R} \) and that \( f(0), f'(0), f''(0) < 0 \). Suppose also that \( f'' \) has the following properties

(i) It is increasing on \([0, \infty)\).
(ii) It has a unique zero at \([0, \infty)\).
(iii) It is unbounded on the interval \([0, \infty)\).

Which of the above three properties hold also for \( f' \)?

(A) \((i)\) only.
(B) \((ii)\) only.
(C) \((iii)\) only.
(D) \((ii)\) and \((iii)\) only.
(E) \((i), (ii)\) and \((iii)\)