1. Peaks and valleys. Find the maximum and minimum values of the function \( f(x) = (x^2 - 4)^8 - 128 \sqrt{4 - x^2} \) over its domain.

**Solution.** For \( x \) to be in the domain of \( f \), the expression inside the radical must be nonnegative, so the domain is \([-2, 2]\). Now we let \( u = \sqrt{4 - x^2} \) for \(-2 \leq x \leq 2\), so \( 0 \leq u \leq 2 \), and substitute:

\[
 f(x) = (-u^2)^8 - 128u = u^{16} - 128u = g(u), 0 \leq u \leq 2.
\]

\[
g'(u) = 16u^{15} - 128 = 0 \text{ if } u^{15} = 8, u = 8^{1/15} = 2^{1/5}. \quad g(0) = 0, g(2) = 2^{16} - 128 \cdot 2 = 2^8 (2^8 - 1) = 65,280, \quad \text{and } g\left(2^{1/5}\right) = 2^{16/5} - 128 \cdot 2^{1/5} = (2^3 - 128) \cdot 2^{1/5} = -120 \cdot 2^{1/5}.
\]

Thus \( \max f = \max g = 65,280 \) and \( \min f = \min g = -120 \cdot 2^{1/5} \sim -137.8438 \).

2. An area. Find the total area of the bounded plane region(s) enclosed by the curves \( y = \frac{1}{2}x - \frac{1}{2}x^{2/3} \) and \( x = y^3 \).

**Solution.** The curves meet when \( y = \frac{1}{2}y^3 - \frac{1}{2} \left( y^3 \right)^{2/3} = \frac{1}{2} \left( y^3 - y^2 \right) \), or \( 0 = y^3 - y^2 - 2y = y(y + 1)(y - 2) \), so \( y = 0, 1 \) or \( 2 \). Thus \( (x, y) = (-1, -1), (0, 0), \) or \( (8, 2) \). \( x = y^3 \iff y = x^{1/3} \). Let \( \Delta(x) = \left( \frac{1}{2} \left( x^{1/3} \right)^3 - \frac{1}{2} \left( x^{1/3} \right)^2 \right) - x^{1/3} = \frac{1}{2}x^{1/3} \left( x^{1/3} + 1 \right) \left( x^{1/3} - 2 \right) \). \(-1 < x < 0 \implies \Delta(x) > 0 \), and \( 0 < x < 8 \implies \Delta(x) < 0 \), so

\[
\text{area} = \int_{-1}^{0} \Delta(x) \, dx + \int_{0}^{8} (-\Delta(x)) \, dx =
\]

\[
= \left[ \frac{1}{4}x^2 - \frac{3}{10}x^{5/3} - \frac{3}{4}x^{4/3} \right]_0^0 + \left[ -\frac{1}{4}x^2 + \frac{3}{10}x^{5/3} + \frac{3}{4}x^{4/3} \right]_0^8 =
\]

\[
= \left( 0 - \left[ \frac{1}{4} + \frac{3}{10} \right] - \frac{3}{4} \right) + \left( \left[ -16 + \frac{48}{5} + 12 \right] - 0 \right) = \frac{1}{5} + \frac{28}{5} = \frac{29}{5} (= 5.8). \]
3. The lost constant. The point $(2, 1)$ is on the curve $x^4 + ky^4 = 16 + k$ no matter what the constant $k$ is. For one particular nonzero choice of $k$, $y'(2) = y''(2)$ along this curve. Find the value of this special choice for $k$.

Solution. For this $k$, let $t = y'(2) = y''(2)$. Differentiate implicitly twice:

$$4x^3 + 4ky^3y' = 0 = 4(3x^2) + 4k\left(3y^2y' \cdot y' + y^3 \cdot y''\right).$$

Plug in $x = 2, y = 1$: $32 + 4kt = 0 = 48 + 4k(3t^2 + t) = 48 + 4kt(3t + 1)$. The first equation gives $4kt = -32$ and $kt = -8$, so the second now gives $0 = 48 - 32(3t + 1)$ or $3t + 1 = 3/2$ and $t = 1/6$. Finally, we get

$$k = \frac{-8}{t} = \frac{-8}{1/6} = -48.$$ 

4. The biggest cylinder. A right circular cone has height 9 and a circular base of radius 6. Find the largest possible volume of a right circular cylinder inscribed in the cone with one end on the base of the cone.

Solution. Place the entire figure in $xyz$-space so that the circular base of the cone is in the $xy$-plane with center $(0, 0, 0)$ and radius 6, while the apex is at $(0, 0, 9)$. The cylinder has radius $r$ $(0 \leq r \leq 6)$, and its top is at height $h$ $(0 \leq h \leq 9)$ where $9r + 6h = 54$, so $h = \frac{54 - 9r}{6} = 9 - \frac{3}{2}r$. As a function of $r$, the volume of the cylinder is $V = \pi r^2 h = \pi r^2 (9 - (3/2)r) = (\pi/2)(18r^2 - 3r^3) = f(r)$. $f'(r) = (\pi/2)(36r - 9r^2) = (9\pi/2)r(4 - r)$ which, for $0 < r < 6$, equals 0 only when $r = 4$. $f(0) = f(6) = 0$ and $f(4) = 48\pi$, which is the largest possible volume.

5. How cool is cool? According to Newton’s law of cooling, the rate at which a cup of coffee cools is proportional to the difference between its temperature and that of the room it is in. A certain cup of coffee cools from $164^\circ$ to $140^\circ$ (all temperatures Fahrenheit) in five minutes, and then from $140^\circ$ to $122^\circ$ in the next five minutes. What is the temperature of the room?

Solution. Let $t$ be the time in minutes after cooling begins, and let $f(t)^\circ$ be the temperature of the coffee at time $t$. Let $c^\circ$ be the temperature of the room. Newton’s law says that $f'(t) = k(f(t) - c)$ for some constant $k$. If $g(t) = f(t) - c$, then $g'(t) = kg(t)$, so $g(t) = \lambda e^{kt}$ for yet another constant $\lambda$. Thus $f(t) = g(t) + c = \lambda e^{kt} + c$. We are given that $164 = f(0) = \lambda + c$, so $\lambda = 164 - c$; that $140 = f(5) = \lambda e^{5k} + c = (164 - c)u + c$ where $u = e^{5k}$; and that $122 = f(10) = \lambda e^{10k} + c = (164 - c)u^2 + c$. Thus $u = \frac{140 - c}{164 - c}$ and $u^2 = \frac{122 - c}{164 - c}$. But also $u^2 = \left(\frac{140 - c}{164 - c}\right)^2$. Equating the two expressions for $u^2$ and multiplying by $(164 - c)^2$ gives $(140 - c)^2 = (122 - c)(164 - c)$ or $c^2 - 280c + 19,600 = c^2 - 286c + 20,008$, so $6c = 408$ and $c = 68$. The temperature of the room is $68^\circ$. 
6. A tricky trig integral. Evaluate the integral

\[ I = \int_{\pi/4}^{\pi/3} \frac{1}{\tan \theta + \cot \theta} \, d\theta. \]

Solution. \[ I = \int_{\pi/4}^{\pi/3} \frac{1}{\tan \theta + \cot \theta} \cdot \frac{\sin \theta \cos \theta}{\sin \theta \cos \theta} \, d\theta = \int_{\pi/4}^{\pi/3} \frac{\sin \theta \cos \theta}{\sin^2 \theta + \cos^2 \theta} \, d\theta = \int_{\pi/4}^{\pi/3} \sin \theta \cos \theta \, d\theta. \]
Substituting \( u = \sin \theta, \) \( du = \cos \theta \, d\theta \) gives
\[ I = \int_{\sqrt{3}/2}^{\sqrt{2}/2} u \, du = \frac{u^2}{2} \bigg|_{\sqrt{3}/2}^{\sqrt{2}/2} = \frac{3}{4} - \frac{2}{4} = \frac{1}{8}. \]

7. A trig series. Determine (proof really needed!) whether the infinite series

\[ \sum_{n=1}^{\infty} \left( 1 - \cos \frac{\pi}{n} \right) \]

converges.

Solution. \( 1 - \cos x \geq 0 \) for all \( x \). Let \( f(x) = x^2 - (1 - \cos x) \). \( f'(x) = 2x - \sin x \) and \( f''(x) = 2 - \cos x > 0 \), so \( f' \) is strictly increasing. \( f'(0) = 0 \), so \( f'(x) > 0 \) for \( x > 0 \) and \( f \) is strictly increasing for \( x \geq 0 \). \( f(0) = 0 \), so \( f(x) > 0 \), that is, \( 1 - \cos x < x^2 \), for all \( x > 0 \). Thus \( 0 \leq 1 - \cos \frac{\pi}{n} < \left( \frac{\pi}{n} \right)^2 \). \[ \sum_{n=1}^{\infty} \left( \frac{\pi}{n} \right)^2 = \pi^2 \sum_{n=1}^{\infty} \frac{1}{n^2}. \] The last sum converges \((p\text{-series}), \) hence by the comparison test so does the given series.

8. Cutting a cone. A cone in \( xyz \)-space has as its cross-section at height \( z \) a circle centered at \((0, 0, z)\) of radius \(|z|\). Consider the solid \( S \) consisting of those points which lie inside the cone, above the \( xy \)-plane, and below the planes \( z = 3 \) and \( z = 2x - 1 \). Set up, but do not evaluate, an integral or iterated double integral or iterated triple integral (or sum of such integrals) whose value is the volume of \( S \). There may be many correct answers; for whatever answer you give, the crucial things to get right are the integrand(s) and all the limits of integration.

Solution. \( S \) consists of those points \((x, y, z)\) that satisfy \( 0 \leq z \leq 3, \frac{z + 1}{2} \leq x \leq z, \) and \( x^2 + y^2 \leq z^2 \). The second of these forces \( \frac{z + 1}{2} \leq z \), so in fact \( z \geq 1 \). Fixing appropriate \( z \) and \( x \), we have \( |y| \leq \sqrt{z^2 - x^2} \). This gives us one possible solution (there are others): the volume of \( S \) is equal to

\[ \int_{1}^{3} \left( \int_{\frac{z+1}{2}}^{z} 2\sqrt{z^2 - x^2} \, dx \right) \, dz. \]