Problem Seminar, Fall 2017-Calculus I

1. (Difficulty 1) Prove that the series
$$\sum_{n=2}^{\infty} \frac{1}{n \log n}$$
diverges.

2. (Difficulty 2) Show that for all $x > 1$,
$$\int_1^x e^{-t^2} dt < \frac{1}{2e}.$$

3. (Difficulty 3) Recall that the n-th harmonic number H_n is defined as
$$H_n = \sum_{k=1}^{n} \frac{1}{k}.$$
Prove that
$$\int_0^1 \frac{1}{x^{n-1} + x^{n-2} + \cdots + x + 1} dx \geq \frac{1}{H_n}.$$

4. (Difficulty 3) Prove that the series
$$\sum_{m=2}^{\infty} \frac{\sin mx}{\log n}$$
converges for all $m \in \mathbb{N}$.

5. (Difficulty 4) Determine if the series
$$\sum_{m=1}^{\infty} \sin mx$$
converges for all $x \in \mathbb{R}$.

6. (Difficulty 4) Let $f : \mathbb{R} \to \mathbb{R}$ be a continuously differentiable 2π-periodic function such that
$$\int_0^{2\pi} f(x) dx = 0.$$
Prove that
$$\int_0^{2\pi} (f'(x))^2 dx \geq \int_0^{2\pi} f^2(x) dx.$$
Hint: Use Fourier Series.

7. (Difficulty 4) Let $f : [0, 1] \to \mathbb{R}$ be a continuous function such that
$$\int_0^1 f(x) dx = \int_0^1 x f(x) dx.$$
Prove that there exists some $c \in (0, 1)$ such that
$$c f(c) = 2 \int_c^0 f(x) dx.$$
Hint: Think about the antiderivative of f.

8. (Difficulty 4) Let $R > 0$. Find the limit
$$\lim_{n\to\infty} \int_0^R \sin(x^n) dx.$$