MATH 3974 PROBLEM SEMINAR HOMEWORK 5, DUE OCTOBER 24

1 (Difficulty 1). Prove that

$$1 - \frac{1}{x} \leq \ln x \leq x - 1,$$

for all $x \geq 1$.

2 (Difficulty 1). Let a, b, c be the side lengths of a triangle with the property that for any positive integer n, the numbers a^n, b^n, c^n can also be the side lengths of a triangle. Prove that the triangle is necessarily isosceles.

3 (Difficulty 2). For any $x, y, z \in \mathbb{R}$, prove that

$$2^{x^2} + 2^{y^2} + 2^{z^2} \geq 2^{xy} + 2^{yz} + 2^{xz}.$$

4 (Difficulty 2). If $a_1 + a_2 + ... + a_n = n$, prove that

$$a_1^4 + a_2^4 + ... + a_n^4 \geq n.$$

5 (Difficulty 2). Prove that if $z \in \mathbb{C}$ satisfies $\mathrm{Re} z < \frac{1}{2}$, then

$$\left| \frac{z}{1-z} \right| < 1.$$

6 (Difficulty 3). Prove that the positive real numbers a, b, c are the side lengths of a triangle if and only if

$$a^2 + b^2 + c^2 < 2\sqrt{a^2b^2 + b^2c^2 + c^2a^2}.$$

7 (Difficulty 3). Let $z_1, .., z_n \in \mathbb{C}$ be so that $|z_1| = |z_2| = ... = |z_n| = 1$. Prove that the number

$$\omega = (z_1 + z_2 + ... + z_n) \left(\frac{1}{z_1} + \frac{1}{z_2} + ... + \frac{1}{z_n} \right)$$

is real and $0 \leq \omega \leq n^2$.

8 (Difficulty 3). Prove that for any three positive numbers a_1, a_2, a_3

$$\frac{a_1^2 + a_2^2 + a_3^2}{a_1^3 + a_2^3 + a_3^3} \geq \frac{a_1^3 + a_2^3 + a_3^3}{a_1^4 + a_2^4 + a_3^4}.$$

9 (Difficulty 4). Show that for all a_1,...,$a_n > 0$ we have

$$\left(\sum_{k=1}^{n}(a_k)^3\right)^2 \leq \left(\sum_{k=1}^{n}(a_k)^2\right)^3.$$

10 (Difficulty 4). Prove that

$$\sqrt{\frac{x}{y+z}}+\sqrt{\frac{y}{x+z}}+\sqrt{\frac{z}{x+y}} > 2$$

for all $x, y, z > 0$.

11 (Difficutly 4). Prove that

$$\frac{a}{b+c}+\frac{b}{c+d}+\frac{c}{d+a}+\frac{d}{a+b} \geq 2$$

12 (Difficulty 5). Let $f : [0,1] \to \mathbb{R}$ be a decreasing function. Prove that for any $a \in (0,1)$

$$a\int_0^1 f(x)\,dx \leq \int_0^a f(x)\,dx.$$

13 (Difficulty 6). Let $(x_n)_{n\geq 1}$ be a sequence of real numbers satisfying

$$x_{n+m} \leq x_n + x_m \qquad \text{for all } n, m \geq 1.$$

Prove that

$$\lim_{n\to\infty}\frac{x_n}{n} \quad \text{exists and is finite.}$$

for all $a, b, c, d > 0$.