Math 3974 Problem Seminar Homework 2

Due September 26, 2017

Problem 1. (Difficulty:1) What is the remainder of 4321^{4321} divided by 13?

Problem 2. (Difficulty:1) Pick two numbers out of $1, 2, \ldots, 100$. What is the probability for the sum to be divisible by 10?

Problem 3. (Difficulty:1) Compute the residue of $1! + 2! + \cdots + 100!$ modulo 15.

Problem 4. (Difficulty:1) Show that any integer that contains each of the nine digits $1, 2, \ldots, 9$ exactly once (for example 359261784) is divisible by 9.

Problem 5. (Difficulty:1) Find the remainder of 123456789 when divided by 11.

Problem 6. (Difficulty:1) There are n books in a library. If books are to be arranged in boxes with 7 books in each box, then 5 books remain. If they are arranged with 9 books in each box, then 3 books remain, and if they are arranged with 11 books in each box, then 7 books remain. What is the smallest possible value for n.

Problem 7. (Difficulty:1) Let p, q be distinct prime numbers. Show that every integer a satisfies the congruence

$$a^{pq-p-q+2} \equiv a \pmod{pq}.$$

Problem 8. (Difficulty:1) Prove that there exists no power of 2 whose decimal presentation ends in the digits 2012.

Problem 9. (Difficulty:2) The number 2^{29} is known to consist of exactly 9 decimal digits, all of which are pairwise distinct. Thus, exactly one of the ten digits $0, 1, 2, \ldots, 9$ is missing. Without using a calculator or brute force hand calculation, determine which digit is missing.

Problem 10. (Difficulty:3) Let an be the sequence defined by $a_1 = 3$, $a_{n+1} = 3^{a_n}$. Let b_n be the remainder when a_n is divided by 100. What is b_{2004}?

Problem 11. (Difficulty:3) Prove that if we write the decimal expansion of $\frac{101}{137}$, it is periodic. (Hint: You need to show that some power of 10 is congruent to 1 modulo 137.)

(Fun fact that I just googled: 137 is the largest prime factor of 123456787654321.)

Problem 12. (Difficulty:3) (Putnam B2) Prove that among any ten consecutive integers at least one is relatively prime to each of the others.

Problem 13. (Difficulty:4) (Putnam) Let n be a positive integer such that $n + 1$ is divisible by 24. Prove that the sum of all the divisors of n is divisible by 24.

Problem 14. (Difficulty:6) (Putnam B2) Find all positive integers n, $k_1, \ldots, k_n$ such that $k_1 + \cdots + k_n = 5n - 4$ and

$$\frac{1}{k_1} + \cdots + \frac{1}{k_n} = 1.$$

Problem 15. (Difficulty:8) (Putnam A3) Let $1, 2, 3, \ldots, 2005, 2006, 2007, 2009, 2012, 2016, \ldots$ be a sequence defined by $x_k = k$ for $k = 1, 2, \ldots, 2006$ and $x_{k+1} = x_k + x_{k-2005}$ for $k \geq 2006$. Show that the sequence has 2005 consecutive terms each divisible by 2006. (Hint: Just think of this sequence modulo 2006. What would be a reasonable interpretation of x_0, x_{-1}, $\ldots$)

Problem 16. (Difficulty:7) (Putnam A4) Prove that for each positive integer n, the number $10^{10^{10^n}} + 10^{10^n} + 10^n - 1$ is not prime. (Hint: Start with $n = 1$.)

Problem 17. (Wolstenholmes theorem) Let $p > 2$ be an odd prime number.

(a) (Difficulty:1) Show that the numerator of

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{(p-1)^2}$$

is divisible by p. (Hint: Here is the formula for sum of squares:

$$1^2 + 2^2 + \cdots + n^2 = \frac{1}{6}n(n+1)(2n+1).$$

Maybe you should consider memorizing it.)

(b) (Difficulty:6) Show that the numerator of

$$1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{p-1}$$

is divisible by p^2. (Hint: Compute $\frac{1}{a} + \frac{1}{p-a}$ and see what happens.)

Problem 18. (Difficulty:9) Show that there exist infinitely many positive integers n such that $n^2 + 1$ divides $n!$.

Problem 19. (Difficulty:9) (Putnam A3) Let p be an odd prime, and let

$$F(n) = 1 + 2n + 3n^2 + \cdots + (p-1)n^{p-2}.$$

Prove that if a and b are distinct integers in $\{0, 1, 2, \ldots, p-1\}$ then $F(a)$ and $F(b)$ are not congruent modulo p. (Hint: The reason I think this problem is very difficult is that, while presented as a number theory question, the key step is some algebraic manipulation of the formula $F(n)$ to write $F(n)$ in a more compact form. This is often the blind spot for problem solving because judging from the appearance, we were already in the number theory mode and forgot that we need to make algebraic manipulations.)