GRE Subject test preparation – Fall 2016

Topic: Abstract Algebra, Linear Algebra, Number Theory, Probability

Linear Algebra

- Standard matrix manipulation to compute the kernel, intersection of subspaces, column spaces, and such. Use matrices to solve system of linear equations.
- The following are equivalent for an $n \times n$-matrix A (acting on $\mathbb{R}^n$)
  - A is invertible, i.e. there exists a matrix B such that $BA = I$ (which is equivalent to exists a matrix B such that $AB = I$).
  - $\mathrm{Ker}A = 0$, i.e. no non-zero vector $v \in \mathbb{R}^n$ such that $Av = 0$.
  - A is surjective, i.e. for every $w \in \mathbb{R}^n$, there exists $v \in \mathbb{R}^n$ such that $Av = w$.
  - $\det A \neq 0$.
  - The columns/rows of A are linearly independent/span $\mathbb{R}^n$.
  - The transpose A^T is invertible.
  - 0 is not an eigenvalue of A
- Kernel/cokernel numerics. Let $f : V \to W$ be a map of finite dimensional linear spaces. Then

  $$\dim V = \dim \mathrm{Ker} f + \dim(\mathrm{Im} f)$$

  Often asked in the form of an inequality, e.g. if $\dim W$ is small, then $\dim \mathrm{Ker} f \geq \dim V - \dim W$.
- Intersection numerics: if W_1 and W_2 are subspaces of a finite dimensional vector space V, then

  $$\dim W_1 + \dim W_2 = \dim(W_1 \cap W_2) + \dim(W_1 + W_2).$$

  Again, often asked in an inequality way: if $\dim W_i$'s are large,

  $$\dim W_1 \cap W_2 \geq \dim W_1 + \dim W_2 - \dim V.$$

- Eigenvalues, Jordan blocks, and Cayley-Hamilton theorem. Let $A \in M_{n\times n}(\mathbb{C})$. Then A is similar to a Jordan normal forms, which is a block diagonal of bunch of Jordan block:

  $$J_r(\lambda) := \begin{pmatrix} \lambda & 1 & 0 & \cdots & 0 & 0 \\ 0 & \lambda & 1 & \cdots & \vdots & \vdots \\ 0 & 0 & \lambda & \cdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda & 1 \\ 0 & 0 & 0 & \cdots & 0 & \lambda \end{pmatrix}$$

  - The characteristic polynomial of A is

    $$\prod_{\lambda \text{ eigenvalues}} (x-\lambda)^{\text{total size of } \lambda\text{-Jordan blocks}}$$

- The minimal polynomial $m_A(x)$ of A (namely, the monic polynomial of minimal degree such that $m_A(A)$) is

$$\prod_{\lambda \text{ eigenvalues}} (x-\lambda)^{\text{maximal size of } \lambda\text{-Jordan blocks}}$$

- So if A satisfies a polynomial with no multiple zeros, then A is diagonal.
- Traces of A is the sum of all eigenvalues (with multiplicity). $\det(A)$ is the product of all eigenvalues (with multiplicity).

Abstract Algebra

- Basic examples of groups: cyclic groups, abelian groups, dihedral groups, S_n, A_n.
- All groups of prime order is cyclic.
- $\mathbb{Z}_p^\times$ is a cyclic group of order $p-1$.
- The generators of $\mathbb{Z}_n$ is all the elements that are coprime to n.

Number Theory

- Modulo arithmetic
- Prime factorization

Probability and statistic

- Probability = number of desired outcomes / total number of outcomes.
- Expectation $E(X)$, standard deviation $\sigma(X) = \sqrt{\mathrm{Var}(x)}$

Linear Algebra

Problem 1 Two distinct solutions $\mathbf{x}_1$ and $\mathbf{x}_2$ can be found to the linear system $A\mathbf{x} = \mathbf{b}$. Which of the following is necessarily true?

(A) $\mathbf{b} = 0$.

(B) A is invertible.

(C) A has more columns than rows.

(D) $\mathbf{x}_1 = -\mathbf{x}_2$.

(E) There exists a solution $\mathbf{x}$ such that $\mathbf{x} \neq \mathbf{x}_1$ and $\mathbf{x} \neq \mathbf{x}_2$.

Problem 2 The solution of the system

$$\begin{aligned} ax + ay - z &= 1 \\ x - ay - az &= -1 \\ ax - y + az &= 1 \end{aligned}$$

is $(x, y, z) = (a, b, a)$. If a is NOT an integer, what is the numerical value of $a+b$?

(A) $-\frac{3}{2}$ (B) -1 (C) 0 (D) $\frac{1}{2}$ (E) 1

Problem 3 Let A, B, and C be real 2×2 matrices, and let 0 denote the 2×2 zero matrix. Which of the following statements is/are true?

I. $A^2 = 0 \;\Rightarrow\; A = 0$

II. $AB = AC \Rightarrow B = C$

III. A is invertible and $A = A^{-1} \Rightarrow A = I$ or $A = -I$.

(A) I only (B) I and III only (C) II and III only (D) III only (E) None of above

Problem 4 If

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^n - \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}^n = \begin{pmatrix} 0 & 6 \\ -6 & 0 \end{pmatrix}$$

then $n =$ (A) -7 (B) -5 (C) 5 (D) 6 (E) 7

Problem 5 If the matrices

$$\begin{pmatrix} 3 & -2 & -2 \\ -1 & 1 & 1 \\ 3 & -1 & -2 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & a0 & \\ -1 & b & 1 \\ 2 & c & -1 \end{pmatrix}$$

are inverses of each other, what is the value of c?

(A) -3 (B) -2 (C) 0 (D) 2 (E) 3

Problem 6 The vectors $\mathbf{v}_1 = (-1, 1, 1)$, $\mathbf{v}_2 = (1, 1, 1)$, $\mathbf{v}_3 = (1, -1, k)$ form a basis for $\mathbb{R}^3$ for all real values of k EXCEPT $k =$

(A) -2 (B) -1 (C) 0 (D) 1 (E) 2

Problem 7 For the matrix

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$$

let r denote its rank and d denote its determinant. What is the value of $r - d$?

(A) -2 (B) -1 (C) 0 (D) 1 (E) 2

Problem 8 If

$$\det \begin{pmatrix} a & b & c \\ k & l & m \\ p & q & r \end{pmatrix} = d$$

then

$$\det \begin{pmatrix} k & 2(a-k) & p+k \\ l & 2(b-l) & q+l \\ m & 2(c-m) & r+m \end{pmatrix} =$$

(A) $-8d$ (B) $-6d$ (C) $-2d$ (D) $2d$ (E) $8d$

Problem 9 For what value of d is the vector $\mathbf{b} = (12, 11, d)^T$ in the column space of this matrix?

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$$

(A) -26 (B) 10 (C) 13 (D) -10 (E) 26

Problem 10 What is the dimension of the following subspace of $\mathbb{R}^5$?

$$\operatorname{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 1 \\ -1 \\ 0 \end{pmatrix} \right\}$$

(A) 1 (B) 2 (C) 3 (D) 4 (E) 5

Problem 11 A square matrix A is said to be **symmetric** if it equals to its own transpose: $A = A^T$. What is the dimension of the subspace $S_{n\times n}(\mathbb{R})$ of real symmetric $n \times n$ matrices in the space of all real $n \times n$ matrices, $M_{n\times n}(\mathbb{R})$?

(A) $\frac{1}{2}n$ (B) $\frac{1}{2}n(n-1)$ (C) $\frac{1}{2}n^2$ (D) $\frac{1}{2}n(n+1)$ (E) $\frac{1}{2}n!$

Problem 12 The set of all points (x, y) in $\mathbb{R}^2$ that satisfy the equation

$$\det \begin{pmatrix} x & y & 1 \\ 0 & y_1 & 1 \\ 1 & y_2 & 1 \end{pmatrix} = 0$$

forms

(A) an ellipse centered at origin

(B) a line with slope $\frac{y_2}{y_1}$

(C) a circle with center $(0, y_1)$ and radius 1

(D) a line with slope $y_2 - y_1$

(E) a parabola with vertex $(1, y_2)$

Problem 13 The linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ that maps $(1,2)$ to $(-1,1)$ and $(0,-1)$ to $(2,-1)$ will map $(1,1)$ to

(A) $(1,2)$ (B) $(1,0)$ (C) $(2,-1)$ (D) $(2,1)$ (E) $(1,1)$

Problem 14 Let $T : \mathbb{R}^5 \to \mathbb{R}^3$ be a linear transformation whose kernel is a three-dimensional subspace of $\mathbb{R}^5$. The set $\{T(\mathbf{x}) : \mathbf{x} \in \mathbb{R}^3\}$ is

(A) the trivial subspace

(B) a line through the origin

(C) a plane through the origin

(D) all of $\mathbb{R}^3$

(E) Cannot be determined from the information given.

Problem 15 Define linear operators S and T on the xy-plane $\mathbb{R}^2$ as follows: S rotates each vector 90° counterclockwise, and T reflects each vector through the y-axis. If ST and TS denote the compositions $S \circ T$ and $T \circ S$, respectively, and I is the identity map, which of the following is true?

(A) $ST = I$ (B) $ST = -I$ (C) $TS = I$ (D) $ST = TS$ (E) $ST = -TS$

Problem 16 Choose a nonzero vector $\mathbf{v} = (a, b, c)^T$ in $\mathbb{R}^3$ and define a linear operator $T : \mathbb{R}^3 \to \mathbb{R}^3$ by the equation $T(\mathbf{x}) = v \times x$, the cross product of $\mathbf{v}$ and $\mathbf{x}$. Then $T(\mathbf{x}) = A\mathbf{x}$ for every $\mathbf{x}$ in $\mathbb{R}^3$ if $A =$

(A) $\begin{pmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{pmatrix}$

(B) $\begin{pmatrix} 1 & -c & b \\ c & 1 & -a \\ -b & a & 1 \end{pmatrix}$

(C) $\begin{pmatrix} 0 & c & -b \\ -c & 0 & a \\ b & -a & 0 \end{pmatrix}$

(D) $\begin{pmatrix} 1 & c & -b \\ -c & 1 & a \\ b & -a & 1 \end{pmatrix}$

(E) $\begin{pmatrix} 0 & a & b \\ a & 1 & c \\ b & c & 0 \end{pmatrix}$

Problem 17 If A is an invertible matrix with an eigenvalue of 3 corresponding to the eigenvector $\mathbf{x}$, which of the following statements must be true?

(A) The matrix A^{-1} has an eigenvalue of $\frac{1}{3}$ corresponding to the eigenvector $\mathbf{x}$.

(B) The matrix A^{-1} has an eigenvalue of $\frac{1}{3}$ corresponding to the eigenvector whose entries are the reciprocals of the entries of $\mathbf{x}$.

(C) The matrix A^2 has an eigenvalue of 3 corresponding to the eigenvector $\mathbf{x}$.

(D) The matrix A^2 has an eigenvalue of 6 corresponding to the eigenvector $2\mathbf{x}$.

(E) The matrix A^2 has an eigenvalue of 3 corresponding to the eigenvector $3\mathbf{x}$.

Problem 18 The eigenvalues of the matrix

$$A = \begin{pmatrix} 2 & b \\ 3 & -1 \end{pmatrix}$$

are -4 and $b - 1$. Find b

(A) 2 (B) 3 (C) 4 (D) 5 (E) 6

Problem 19 The complex matrix

$$A = \begin{pmatrix} 2 & 2+i \\ 2-i & 6 \end{pmatrix}$$

has which one of the following as an eigenvalue?

(A) -1 (B) 3 (C) 7 (D) i (E) $1+i$

Abstract Algebra

Problem 1 How many generators does the group $(\mathbb{Z}_{24}, +)$ have?

(A) 2 (B) 6 (C) 8 (D) 10 (E) 12

Problem 2 which one of the following groups is cyclic?

(A) $\mathbb{Z}_2 \times \mathbb{Z}_4$ (B) $\mathbb{Z}_2 \times \mathbb{Z}_6$ (C) $\mathbb{Z}_3 \times \mathbb{Z}_4$ (D) $\mathbb{Z}_3 \times \mathbb{Z}_6$ (E) $\mathbb{Z}_4 \times \mathbb{Z}_6$

Problem 3 If G is a group of order 12, then G must have a subgroup of all of the following order EXCEPT

(A) 2 (B) 3 (C) 4 (D) 6 (E) 12

Problem 4 How many subgroups does the group $\mathbb{Z}_3 \times \mathbb{Z}_1 6$ have?

(A) 6 (B) 10 (C) 12 (D) 20 (E) 24

Problem 5 If $S = \{a \in \mathbb{R}^+ : a \neq 1\}$, with the binary operation $\bullet$ defined by the equation $a \bullet b = a^{\log b}$ (where $\log b = \log_e b$), then $(S, \bullet)$ is a group. What is the inverse of $a \in S$?

(A) $\frac{1}{e \log a}$ (B) $\frac{e}{\log a}$ (C) $e^{-\log a}$ (D) $e^{\log(1/a)}$ (E) $e^{1/\log a}$

Problem 6 Which of the following are subgroups of $\mathrm{GL}(2, \mathbb{R})$, the group of invertible 2×2 matrices (with real entries) under matrix multiplication?

I. $T = \{A \in \mathrm{GL}(2, \mathbb{R}) : \det A = 2\}$

II. $U = \{A \in \mathrm{GL}(2, \mathbb{R}) : A \text{ is upper triangular}\}$

III. $T = \{A \in \mathrm{GL}(2, \mathbb{R}) : \mathrm{tr}(A) = 0\}$

Note: $\mathrm{tr}(A)$ denotes the *trace* of A.

(A) I and II only (B) II only (C) II and III only

(D) III only (E) I and III only

Problem 7 Let H be the set of all group homomorphisms $\Phi : \mathbb{Z}_3 \to \mathbb{Z}_6$. How many functions does H contain?

(A) 1 (B) 2 (C) 3 (D) 4 (E) 6

Problem 8 Let G be a group of order 9, and let e denote the identity of G. Which one of the following statements about G CANNOT be true?

(A) There exists an element x in G such that $x \neq e$ and $x^{-1} = x$.

(B) There exists an element x in G such that $x \neq e$ and $x^2 = x^5$.

(C) There exists an element x in G such that $\langle x \rangle$ has order 3.

(D) G is cyclic.

(E) G is abelian.

Problem 9 Let R be a ring; an element x in R is said to be an idempotent if $x^2 = x$. How many idempotent elements does the ring $\mathbb{Z}_{20}$ contain?

(A) 2 (B) 4 (C) 5 (D) 8 (E) 10

Problem 10 Which of the following rings are integral domains?

I. $\mathbb{Z} \times \mathbb{Z}$

II. $\mathbb{Z}_p$, where p is a prime

III. $\mathbb{Z}_{p^2}$, where p is a prime

(A) I and II only (B) II only (C) II and III only

(D) III only (E) I and III only

Problem 11 Which of the following is a subfield of $\mathbb{C}$?

I. $K_1 = \left\{a + b\sqrt{\frac{2}{3}} : \ a, b \in \mathbb{Q}\right\}$

II. $K_1 = \left\{a + b\sqrt{2} : \ a, b \in \mathbb{Q} \text{ and } ab < \sqrt{2}\right\}$

III. $K_3 = \left\{a + bi : \ a, b \in \mathbb{Z} \text{ and } i = \sqrt{-1}\right\}$

(A) I only (B) I and II only (C) III only (D) I and III only

(E) None of the K_i are subfields of $\mathbb{C}$.

Problem 12 Let pq be distinct primes. How many (mutually nonisomorphic) abelian groups are there of order p^2q^4?

(A) 6 (B) 8 (C) 10 (D) 12 (E) 16

Number Theory

Problem 1 There is only one integer x, between 100 and 200 such that integer pair (x, y) satisfies the equation $42x + 55y = 1$. What is the value of x in this integer pair?

(A) 127 (B) 148 (C) 158 (D) 167 (E) 183

Problem 2 Let L be the least common multiple of 1001 and 10101. What's the sum of the digits of L? (All numbers are written in their usual decimal representations.)

(A) 6 (B) 11 (C) 17 (D) 22 (E) 33

Problem 3 Let x_1 and x_2 be the two smallest positive integers for which the following statement is true: "$85x - 12$ is a multiple of 19." Then $x_1 + x_2 =$

(A) 19 (B) 27 (C) 31 (D) 38 (E) 47

Problem 4 If x, y, and z are positive integers such that $4x + 5y - 2z$ is divisible by 13, then which one of the following must also be divisible by 13?

(A) $x+13y-z$ (B) $6x-10y-z$ (C) $x-y-2z$ (D) $-7x+12y+3z$ (E) $-5x+3y-4z$

Problem 5 When expressed in its usual decimal notation, the number 100! (that is, 100 factorial) ends in how many consecutive zeros?

(A) 20 (B) 24 (C) 30 (D) 32 (E) 50

Probability and counting

Problem 1 In a State Lottery, players choose 6 distinct numbers from among the integers 1 through 51. The jackpot is awarded if the 6 numbers selected match the 6 numbers drawn. What is the minimal number of tickets someone would need to purchase in order to guarantee winning?

(A) 720 (B) 18,009,460 (C) 377,149,517 (D) 12,966,811,200 (E) 17,596,287,801

Problem 2 A fair die is tossed twice. About how many ties would you expect to roll 3 or greater?

(A) 2 (B) $\frac{3}{2}$ (C) 1 (D) $\frac{1}{2}$ (E) 0

Problem 3 At a banquet, 9 women and 6 men are to be seated in a row of 15 chairs. If the entire seating arrangement is to be chosen at random, what is the probability that all of the men will be seated next to each other in 6 consecutive positions?

(A) $\dfrac{1}{\binom{15}{6}}$ (B) $\dfrac{6!}{\binom{15}{6}}$ (C) $\dfrac{10!}{15!}$ (D) $\dfrac{6!9!}{14!}$ (E) $\dfrac{6!10!}{15!}$

Problem 4 Suppose X is a discrete random variable on the set of positive integers such that for each positive integer n, the probability that $X = n$ is $\frac{1}{2^n}$. If Y is a random variable with the same probability distribution and X and Y are independent, what is the probability that the value of at least one of the variables X and Y is greater than 3?

(A) $\dfrac{1}{64}$ (B) $\dfrac{15}{64}$ (C) $\dfrac{1}{4}$ (D) $\dfrac{3}{8}$ (E) $\dfrac{4}{9}$

Problem 5 A fair coin is to be tossed 100 times, with each toss resulting in a head or a tail. If H is the total number of heads and T is the total number of tails, which of the following events has the greatest probability?

(A) $H = 50$ (B) $T \geq 60$ (C) $51 \leq H \leq 55$

(D) $H \geq 48$ and $T \geq 48$ (E) $H \leq 5$ or $H \geq 95$

Problem 6 A university's mathematics department has 10 professors and will offer 20 different courses next semester. Each professor will be assigned to teach exactly 2 of the courses, and each course will have exactly one professor assigned to teach it. If any professor can be assigned to teach any course, how many different assignments of the 10 professors to the 20 courses are possible?

(A) $\dfrac{20!}{2^{10}}$ (B) $\dfrac{10!}{2^9}$ (C) $10^{20} - 2^{10}$ (D) $10^{20} - 100$ (E) $\dfrac{20!10!}{2^{10}}$