MATH 3974 PROBLEM SEMINAR HOMEWORK 4, DUE OCTOBER 18

1. (Difficulty 1) Let A be an $n \times n$ matrix such that $A^3 = A + I_n$. Prove that $A + I_n$ is invertible.

2. (Difficulty 1) Let A be an $n \times n$ matrix whose entries are odd integers. Show that $\det A$ is divisible by 2^{n-1}.

3. (Difficulty 1) Let A and B be 2×2 matrices with $\det A = \det B = 1$. Prove that

$$tr\,(AB) - tr\,(A)\,tr\,(B) + tr\left(AB^{-1}\right) = 0.$$

4. (Difficulty 2) Let A be a $(2n+1) \times (2n+1)$ matrix. Prove that $\det\left(A - A^t\right) = 0$.

5. (Difficulty 2) Let A be an $n \times n$ matrix so that $\det\left(A + I_n\right) \geq 0$. Prove that $\det\left(A^3 + I_n\right) \geq 0$ as well.

6. (Difficulty 2) Given two $n \times n$ matrices A and B for which $AB = A + B$, prove that $AB = BA$.

7. (Difficulty 3) Let A and B be 3×3 matrices with real entries such that

$$\det A = \det B = \det\left(A + B\right) = \det\left(A - B\right) = 0.$$

Prove that $\det\left(A + xB\right) = 0$ for any real number x.

8. (Difficulty 4) Let A and B be two $n \times n$ matrices that do not commute and for which there exist nonzero real numbers p, q, r so that $pAB + qBA = I_n$ and $A^2 = rB^2$. Prove that $p = q$.

9. (Difficulty 4) Let A, B be $n \times n$ matrices so that $3AB - 2BA = I_n$. Prove that $\det\left(AB - BA\right) = 0$.

10. (Difficulty 5) Let A be an $n \times n$ matrix whose entry on the i-th row and j-th column is

$$\frac{1}{\min\left(i, j\right)},$$

for all $i, j = 1, .., n$. Compute $\det A$.

11. (Difficulty 5) Let $a = \frac{2\pi}{n}$. Prove that the matrix

$$\begin{pmatrix} 1 & 1 & .... & 1 \\ \cos a & \cos 2a & .... & \cos na \\ . & . & . & . \\ . & . & . & . \\ \cos(n-1)a & \cos 2(n-1)a & .... & \cos n(n-1)a \end{pmatrix}$$

is invertible.

12. (Difficulty 6) Let A and B be 2×2 matrices with integer entries, such that $AB = BA$ and $\det B = 1$. Prove that if $\det\left(A^3 + B^3\right) = 1$, then $A^2 = O$.

13. (Difficulty 6) Let A and B be 2×2 matrices so that $\det(AB + BA) \leq 0$. Prove that $\det\left(A^2 + B^2\right) \geq 0$.