MATH 3974 PROBLEM SEMINAR HOMEWORK 3, DUE OCTOBER 4

1. (Difficulty 1) Let $f : (1, \infty) \to \mathbb{R}$ be defined by $f(x) = \ln\left(2x^2 - 3x + 1\right)$. Compute $f^{(n)}(x)$ for all $x > 1$.

2. (Difficulty 1) Compute $(f \circ f \circ ... \circ f)(x)$ for the function $f(x) = 2x + 1$.

3. (Difficulty 2) Prove that if $a_1 \geq a_2 \geq ... \geq a_n \geq 0$ then

$$a_1^2 + 3a_2^2 + ... + (2n-1)\, a_n^2 \leq (a_1 + a_2 + ... + a_n)^2 .$$

4. (Difficulty 2) Prove that for all $n \geq 0$, the polynomial $(X-1)^{2n+1} + (-1)^{n+1} X^{n+2}$ is divisible by $X^2 - X + 1$.

5. (Difficulty 2) Prove that $|\sin(nx)| \leq n |\sin x|$ for any real number x and positive integer n.

6. (Difficulty 3) Find all functions $f : \mathbb{R} \to \mathbb{R}$ continuous in $x = 0$ such that $f(2x) = f(x) + x$ for all $x \in \mathbb{R}$.

7. (Difficulty 3) Prove that

$$\frac{1}{1^2} + \frac{1}{2^2} + ... + \frac{1}{n^2} < 2.$$

8. (Difficulty 4) Show that if $m > n$ are nonnegative integers, then

$$\int_0^{\pi} \cos^n(x) \cdot \cos(mx)\, dx = 0.$$

9. (Difficulty 5) Prove that for $n \geq 6$ the equation

$$\frac{1}{x_1^2} + ... + \frac{1}{x_n^2} = 1$$

has at least one solution $x_1, ..., x_n$ of positive integers.

10. (Difficulty 5) Prove that the sequence $(a_n)_{n \geq 2}$ defined by

$$a_n = \frac{n}{\sqrt[n]{n!}}$$

is increasing.

11. (Difficulty 6) Define the sequence $(x_n)_{n\geq 1}$ by $x_1 = 1$ and for $n \geq 1$,

$$x_{n+1} = \frac{n}{x_n} + \frac{x_n}{n}.$$

Compute

$$\lim_{n\to\infty} \frac{x_n}{\sqrt{n}}.$$

12. (Difficulty 7) Let $a_0 = \frac{5}{2}$ and $a_k = a_{k-1}^2 - 2$ for $k \geq 1$. Compute

$$\prod_{k=1}^{\infty} \left(1 - \frac{1}{a_k}\right)$$

in closed form.

13. (Difficulty 7) Prove that for any $x \in (0, \pi)$, and any $n \geq 1$ the following inequality holds:

$$\frac{\sin x}{1} + \frac{\sin 2x}{2} + ... + \frac{\sin nx}{n} > 0.$$