Math 3974 Problem Seminar Homework 1

Due September 13, 2016

Problem 1.1. (Difficulty:1) Compute the sum

$$\frac{1}{1\cdot 2\cdot 3}+\frac{1}{2\cdot 3\cdot 4}+\cdots+\frac{1}{98\cdot 99\cdot 100}.$$

Problem 1.2. (Difficulty:1) Compute the sum

$$1\cdot 2\cdot 3+2\cdot 3\cdot 4+\cdots+98\cdot 99\cdot 100.$$

Problem 1.3. (Difficulty:1) Compute the sum

$$\sum_{n=1}^{N} n\cdot n!$$

Problem 1.4. (Difficulty:2) Compute the sum

$$\frac{1}{\sqrt{2}+1}+\frac{1}{\sqrt{3}+\sqrt{2}}+\cdots+\frac{1}{\sqrt{99}+\sqrt{100}}.$$

Problem 1.5. (Difficulty:2) Find the product

$$\prod_{n=2}^{2016}\left(1-\frac{1}{n^2}\right).$$

Problem 1.6. (Difficulty:3) Evaluate

$$\sum_{n=1}^{2016}\frac{1}{\sqrt[3]{n^2}+\sqrt[3]{n(n+1)}+\sqrt[3]{(n+1)^2}}.$$

Problem 1.7. (Difficulty:3) Given that $a_1, a_2, \ldots, a_n$ are positive numbers in arithmetic progression, show that

$$\frac{1}{a_1a_2}+\frac{1}{a_2a_3}+\cdots+\frac{1}{a_{n-1}a_n}=\frac{n-1}{a_1a_n}$$

and

$$\frac{1}{\sqrt{a_1}+\sqrt{a_2}}+\cdots+\frac{1}{\sqrt{a_{n-1}}+\sqrt{a_n}}=\frac{n-1}{\sqrt{a_1}+\sqrt{a_n}}.$$

Problem 1.8. (Difficulty:3) Evaluate

$$\sum_{k=0}^{r}(-1)^k\binom{n}{k}$$

for $0\le r\le n$.

Problem 1.9. (Difficulty:3) Evaluate the infinite series

$$\sum_{n=0}^{\infty}\frac{n}{n^4+n^2+1}.$$

Hint: x^4+x^2+1 is not an irreducible polynomial. Remember this!

Problem 1.10. (Difficulty:3) (Putnam 1977B1) Evaluate the infinite product

$$\prod_{n=2}^{\infty} \frac{n^3-1}{n^3+1}.$$

Problem 1.11. (Difficulty:3) Find the value of the following infinite tower of exponents:

$$\sqrt{2}^{\sqrt{2}^{\sqrt{2}^{\sqrt{2}^{\cdots}}}}$$

interpreted as the limit of the infinite sequence 1, $\sqrt{2}$, $\sqrt{2}^{\sqrt{2}}$, $\sqrt{2}^{\sqrt{2}^{\sqrt{2}}}$, $\cdots$. Be rigorous about your argument.

Problem 1.12. (Difficulty:4) If $x \neq 0$, prove that

$$\frac{\sin x}{x} = \prod_{n=1}^{\infty} \cos\left(\frac{x}{2^n}\right).$$

Problem 1.13. (Difficulty:4) Evaluate the sum

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{1^3+2^3+3^3+\cdots+n^3}}.$$

Hint: First compute the sum $1^3+2^3+3^3+\cdots+n^3$ for small n, did you see some pattern?

Problem 1.14. (Difficulty:4) Find the limit

$$\lim_{n\to\infty} \prod_{k=0}^{n} \left(1+\frac{1}{3^{2^k}}\right).$$

Problem 1.15. (Difficulty:6) Determine the whether the following value is finite

$$\sqrt{1^2+\sqrt{2^2+\sqrt{3^2+\sqrt{\cdots}}}}.$$

Problem 1.16. (Difficulty:8) (Putnam 2015A3) Let $a_0 = \frac{5}{2}$ and $a_k = a_{k-1}^2 - 2$ for $k \geq 1$. Compute the following in a closed form

$$\prod_{k=0}^{\infty} \left(1-\frac{1}{a_k}\right).$$

Hint: Try to get the expression of a_k first. Try out a few terms and you can see the pattern.