SOLUTIONS TO 2017 UCONN UNDERGRADUATE CALCULUS COMPETITION EXAM

Thursday 6 April 2017, 6:30-8:00 p.m.

Please show enough of your work so your line of reasoning will be clear. Numerical answers will receive no credit if they are not adequately supported. Calculators are welcome, but unlikely to be very useful. Have fun, and good luck!

1. **Find the polynomial.** Find a cubic (that is, third degree) polynomial $p(x) = x^3 + ax^2 + bx + c$ such that the graph of p has a local (or relative) maximum at $(x, y) = (-3, 10)$ and a point of inflection when $x = -5/3$.

   Solution. $p'(x) = 3x^2 + 2ax + b$ and $p''(x) = 6x + 2a$. From $p''(-5/3) = 0$ we get $-10 + 2a = 0$ or $a = 5$. From $p'(-3) = 0$ we get $27 - 30 + b = 0$ or $b = 3$. From $p(-3) = 10$ we get $-27 + 45 - 9 + c = 10$ or $c = 1$. Thus the desired polynomial is $p(x) = x^3 + 5x^2 + 3x + 1$.

2. **The biggest inscribed triangle.** In the xy-plane, a triangle is inscribed in the closed curve consisting of the semicircle whose equation is $x^2 + y^2 = 1, x \geq 0$, and the vertical segment consisting of points on the y-axis for which $-1 \leq y \leq 1$, in such a way that one of its sides is a chord parallel to the y-axis and the remaining vertex lies on the y-axis. Find (with justification) the maximum possible area of such a triangle.

   Solution. Let the base of the triangle be the "vertical" chord, going from $(\sqrt{1-\alpha^2}, -\alpha)$ to $(\sqrt{1-\alpha^2}, \alpha)$ for some α in $[0, 1]$. The remaining vertex is $(0, \beta)$ for some β in $[-1, 1]$. The altitude of the triangle is $\sqrt{1-\alpha^2}$, and so its area is $f(\alpha) = (1/2)(2\alpha)\sqrt{1-\alpha^2}) = \alpha\sqrt{1-\alpha^2}$ regardless of the choice of β. Thus the maximum possible area is the maximum possible value of $f(\alpha)$ for $0 \leq \alpha \leq 1$. For $0 < \alpha < 1$,
$$f'(a) = \sqrt{1-\alpha^2} + \alpha \cdot \frac{-\alpha}{\sqrt{1-\alpha^2}} = \frac{1-2\alpha^2}{\sqrt{1-\alpha^2}} \ .$$
   Thus $f'(\alpha) = 0$ only if $\alpha = \sqrt{1/2}$. Since $f(0) = f(1) = 0$ and $f(\alpha) \geq 0$ for all α in $[0, 1]$, we see that the maximum we seek must be $f(\sqrt{1/2}) = 1/2$.

3. **A limit of sums.** For $n = 1, 2, \ldots$ let

$$x_n = \frac{n+1}{9n^2 + (n+1)^2} + \frac{n+2}{9n^2 + (n+2)^2} + \cdots + \frac{9n}{9n^2 + (9n)^2} = \sum_{k=n+1}^{9n} \frac{k}{9n^2 + k^2}.$$

Thus, for instance,
$x_1 = \dfrac{2}{9+2^2} + \dfrac{3}{9+3^2} + \cdots + \dfrac{9}{9+9^2}$ and $x_2 = \dfrac{3}{36+3^2} + \dfrac{4}{36+4^2} + \cdots + \dfrac{18}{36+18^2}$.
Compute $\lim_{n\to\infty} x_n$, expressing your answer in as simple a form as possible..

Solution. Note that $x_n = \displaystyle\sum_{k=n+1}^{9n} \frac{k/n}{9 + (k/n)^2} \cdot \frac{1}{n}$ is the n^{th} Riemann sum for, and therefore converges to,

$$\int_1^9 \frac{x}{9 + x^2}\, dx = \frac{1}{2} \ln{(9 + x^2)}|_1^9 = \frac{1}{2}(\ln 90 - \ln 10) = \frac{1}{2} \ln 9 = \ln 3.$$

4. **A tricky integral.** Evaluate the definite integral

$$I = \int_1^4 \frac{x}{x\sqrt{x} + 8}\, dx\,.$$

Again, express your answer in as simple a form as possible.

Solution. The substitution $x = u^2, dx = 2udu$ $(1 \leq u \leq 2)$ converts I to

$$I = \int_1^2 \frac{2u^3}{u^3+8}\, du = \int_1^2 \frac{(2u^3+16)-16)}{u^3+8} = \int_1^2 \left(2 - \frac{16}{(u+2)(u^2-2u+4)}\right) du =$$

$$= 2 - \int_1^2 \left(\frac{A}{u+2} + \frac{Bu+C}{u^2-2u+4}\right) du$$

where A, B, C are constants such that $16 = A(u^2 - 2u + 4) + (Bu + C)(u + 2) = (A + B)u^2 + (-2A + 2B + C)u + (4A + 2C)$. The u^2 term gives $B = -A$, then the u term gives $C = 4A$, and then the constant term gives $12A = 16$, that is, $A = 4/3$, so $B = -4/3$ and $C = 16/3$. Thus $I = 2 - J - K$ where $J = \displaystyle\int_1^2 \frac{4/3}{u+2}\, du = (4/3) \ln{(u+2)}|_1^2 = (4/3) \ln{(4/3)}$ and $K = \displaystyle\int_1^2 \frac{-(4/3)u + 16/3}{u^2 - 2u + 4}\, du = \int_1^2 \frac{-(2/3)(2u-2)}{u^2-2u+4}\, du + \int_1^2 \frac{4}{u^2-2u+4}\, du =$
$= (-2/3) \ln{(u^2 - 2u + 4)}|_1^2 + \displaystyle\int_1^2 \frac{4/3}{\left(\frac{u-1}{\sqrt{3}}\right)^2 + 1}\, du = -(2/3) \ln{(4/3)} + \frac{4}{\sqrt{3}} \arctan\left(\frac{u-1}{\sqrt{3}}\right) |_1^2 =$
$= -(2/3) \ln{(4/3)} + \dfrac{4}{\sqrt{3}} \left(\dfrac{\pi}{6} - 0\right)$. Finally, then,

$$I = 2 - (4/3) \ln{(4/3)} + (2/3) \ln{(4/3)} - \frac{2\pi\sqrt{3}}{9} = 2 - (2/3) \ln{(4/3)} - \frac{2\pi\sqrt{3}}{9}.$$

5. **The root of the matter.** Give a precise meaning to the expression

$$\rho = \sqrt{6 + \sqrt{6 + \sqrt{6 + \sqrt{6 + \dots}}}}$$

as a limit, show that the limit exists, and find its value.

Solution. To begin at the end, if ρ does have a numerical value, clearly it satisfies $\rho = \sqrt{6+\rho}$, that is, $\rho^2 = 6 + \rho$, or $\rho^2 - \rho - 2 = 0$, so (since evidently $\rho \geq 0$) $\rho = 3$.

Now for the technical stuff. Define a sequence $(x_n)_{n=1}^{\infty}$ by $x_1 = \sqrt{6}$ and $x_{n+1} = \sqrt{6 + x_n}$ for $n \geq 1$. Clearly $x_1 = \sqrt{6} < 3$ and $x_1 = \sqrt{6} < \sqrt{6 + x_1} = x_2$, and an easy induction now shows that for all $n \geq 1$, $x_n < 3$ and $x_n < x_{n+1}$. Thus (x_n) is an increasing sequence of positive numbers all less than 3, so converges to a positive limit ρ that is no more than 3. Then $\rho = \lim_{n\to\infty} x_n = \lim_{n\to\infty} \sqrt{6 + x_{n-1}} = \sqrt{6 + \lim_{n\to\infty} x_{n-1}} = \sqrt{6+\rho}$, which as we have seen implies that $\rho = 3$.

6. **A cap in a cone.** A right circular cone $\mathcal{C}$ has altitude 40 inches and a circular base of radius 30 inches. A sphere $\mathcal{S}$ is inscribed in $\mathcal{C}$ so that it is tangent to $\mathcal{C}$ at the center of its base and at all the points of a circle whose plane is parallel to the base. $\mathcal{S}$ divides $\mathcal{C}$ into three solid regions: the interior of $\mathcal{S}$, the region "below" $\mathcal{S}$ and "above" the base of $\mathcal{C}$, and the region $\mathcal{R}$ "above" $\mathcal{S}$. Compute the volume of $\mathcal{R}$.

Solution. Let's situate the cone $\mathcal{C}$ in (xyz)-space so that its base is the disc $D = \{(x, y, 0) : x^2 + y^2 \leq 30^2\}$ in the (xy)-plane and its apex is at $(0, 0, 40)$. The sphere $\mathcal{S}$ has the equation $x^2 + y^2 + (z - \rho)^2 = \rho^2$ for some radius ρ. The cross-section of this figure in the (xz)-plane is a triangle T with vertices at $(-30, 0), (30, 0)$, and $(0, 40)$, and a circle W with equation $x^2 + (z - \rho)^2 = \rho^2$. W is tangent to T at three points: $(0, 0)$ and points $(\pm x_0, z_0)$ on the sides of T. The right-hand side of T is the segment L of slope -4/3 going from $(30, 0)$ to $(0, 40)$ whose equation is $40x + 30z = 1200$.

The radius of W whose end is the point of tangency on L has length ρ and slope 3/4, so goes from $(0, \rho)$ to $(x_0, z_0) = (0 + (4/5)\rho, \rho + (3/5)\rho) = ((4/5)\rho, (8/5)\rho)$. Since this point lies on L, we have $1200 = 40(4/5)\rho + 30(8/5)\rho = 80\rho$, so $\rho = 15$ and W has equation $x^2 + (z - 15)^2 = 15^2$, its "top" is at $(0, 30)$, and its points of tangency with T are at $(\pm 12, 24)$.

So the sphere $\mathcal{S}$ has equation $x^2 + y^2 + (z - 15)^2 = 15^2$ or $x^2 + y^2 = 30z - z^2$ and is tangent to the cone $\mathcal{C}$ along the circle at height $z = 24$. The volume of the portion of $\mathcal{C}$ above height $z = 24$ is $V = (1/3)(\pi \cdot 12^2)(40 - 24) = 768\pi$. At height z, $24 \leq z \leq 30$, the cross-section of $\mathcal{S}$ is a circle of radius $\sqrt{30z - z^2}$, so bounds an area of $\pi(30z - z^2)$. Thus the volume inside $\mathcal{S}$ and above height $z = 24$ is $V' = \int_{24}^{30} \pi(30z - z^2)\,dz = \pi \cdot (15z^2 - (1/3)z^3)\,|_{24}^{30} = \pi(4,500 - 4,032) = 468\pi$. Finally, the volume we seek is the difference

$$V - V' = 768\pi - 468\pi = 300\pi \text{ cubic inches.}$$

7. **Let's get to the bottom of this.** Let $f(x, y) = x^2 + y^2 - 4\sin(xy)$.

   (a) Assuming that $f(x, y)$ actually attains a minimum value as (x, y) ranges over the plane, find this minimum value.

   (b) Prove that $f(x, y)$ actually *does* attain a minimum value (which was to be computed in part (a)).

   Solution. We do the parts separately

   (a) We require $0 = f_x = 2x - 4y\cos(xy)$ and $0 = f_y = 2y - 4x\cos(xy)$. Subtracting gives $2(x - y)[1 + 2\cos(xy)] = 0$ while adding gives $2(x + y)[1 - 2\cos(xy)] = 0$. The two bracketed expressions cannot both be 0, so either $x - y = 0$ or $x + y = 0$.

   If $x - y = 0$, let $x = y = t$ and $z = t^2 \geq 0$. Then $f(x, y) = 2t^2 - 4\sin(t^2) = 2z - 4\sin z = g(z)$. $g(0) = 0$ and $g'(z) = 2 - 4\cos z = 0$ if $\cos z = 1/2$, so $z = \pi/3 + 2k\pi$ or $z = (5\pi)/3 + 2k\pi$ for k a nonnegative integer. $g(\pi/3) = (2\pi)/3 - 2/\sqrt{3} < 0$ while $g(z) > 0$ for all the other nonzero choices of z, so the minimum value of $g(z)$ is $f(\sqrt{\pi/3}, \sqrt{\pi/3}) = (2\pi)/3 - 2\sqrt{3} < 0$.

   If $x + y = 0$, let $x = -y = t$ and $z = t^2 \geq 0$. Then $f(x, y) = 2t^2 + 4\sin(t^2) = 2z + 4\sin z = h(z)$. $h(0) = 0$ and $h'(z) = 2 + 4\cos z = 0$ if $z = -1/2$. The nonzero choices of z are all at least $(2\pi)/3$, so $h(z) \geq (4/3)\pi - 4 > 0$.

   So the minimum value of $f(x, y)$ is $(2/3)\pi - 2\sqrt{3}$, when $(x, y) = (\sqrt{\pi/3}, \sqrt{\pi/3})$.

   (b) Let $D = \{(x, y) : |x| \leq 2, |y| \leq 2\}$, a closed bounded set, so f assumes a minimum value m on D, and $m \leq f(0, 0) = 0$. If $(x, y) \notin D$ then $x^2 > 4$ or $y^2 > 4$ (or both) so $f(x, y) > 0 \geq 0$. Thus $f(x, y)$ has a minimum value m, which we computed in part (a).

8. **The big race.** Laura is competing in the 100-yard dash. For at least the first 60 yards, her (varying) speed in yards per second is proportional to the cube root of the distance she has run in yards. When she has run 16 yards, her speed is 6 yards per second. How many seconds does it take Laura to run the first 54 yards? Justify your answer.

   Solution. Let y denote the number of yards Laura has run after t seconds have elapsed, so $y = 0$ when $t = 0$ and $y > 0$ when $t > 0$. By assumption there is a constant $c > 0$ such that $y' = cy^{1/3}$, at least as long as $y \leq 60$. Since $y' = 6$ when $y = 16$, we have $c = 6 \cdot 16^{-1/3} = 3 \cdot 2^{-1/3}$. Integrating $y'/y^{1/3} = c$ with respect to t gives $(3/2)y^{2/3} = ct$ (since $y(0) = 0$). When $y = 54$, $y^{2/3} = 9 \cdot 2^{2/3}$, so $t = (3/2) \cdot 9 \cdot 2^{2/3}/(3 \cdot 2^{-1/3}) = 9$, which is the answer.