2017 UCONN UNDERGRADUATE CALCULUS COMPETITION

Thursday 6 April 2017, 6:30-8:00 p.m.

Please show enough of your work so your line of reasoning will be clear. Numerical answers will receive no credit if they are not adequately supported. Calculators are welcome, but unlikely to be very useful. Have fun, and good luck!

1. **Find the polynomial.** Find a cubic (that is, third degree) polynomial $p(x) = x^3 + ax^2 + bx + c$ such that the graph of p has a local (or relative) maximum at $(x, y) = (-3, 10)$ and a point of inflection when $x = -5/3$.

2. **The biggest inscribed triangle.** In the xy-plane, a triangle is inscribed in the closed curve consisting of the semicircle whose equation is $x^2 + y^2 = 1, x \geq 0$, and the vertical segment consisting of points on the y-axis for which $-1 \leq y \leq 1$, in such a way that one of its sides is a chord parallel to the y-axis and the remaining vertex lies on the y-axis. Find (with justification) the maximum possible area of such a triangle.

3. **A limit of sums.** For $n = 1, 2, \ldots$ let

$$x_n = \frac{n+1}{9n^2 + (n+1)^2} + \frac{n+2}{9n^2 + (n+2)^2} + \cdots + \frac{9n}{9n^2 + (9n)^2} = \sum_{k=n+1}^{9n} \frac{k}{9n^2 + k^2}.$$

   Thus, for instance,
   $x_1 = \dfrac{2}{9+2^2} + \dfrac{3}{9+3^2} + \cdots + \dfrac{9}{9+9^2}$ and $x_2 = \dfrac{3}{36+3^2} + \dfrac{4}{36+4^2} + \cdots + \dfrac{18}{36+18^2}$.
   Compute $\lim_{n\to\infty} x_n$, expressing your answer in as simple a form as possible..

4. **A tricky integral.** Evaluate the definite integral

$$I = \int_1^4 \frac{x}{x\sqrt{x} + 8}\, dx\,.$$

   Again, express your answer in as simple a form as possible.

5. **The root of the matter.** Give a precise meaning to the expression

$$\rho = \sqrt{6 + \sqrt{6 + \sqrt{6 + \sqrt{6 + \ldots}}}}$$

   as a limit, show that the limit exists, and find its value.

Continued on other side.

6. **A cap in a cone.** A right circular cone $\mathcal{C}$ has altitude 40 inches and a circular base of radius 30 inches. A sphere $\mathcal{S}$ is inscribed in $\mathcal{C}$ so that it is tangent to $\mathcal{C}$ at the center of its base and at all the points of a circle whose plane is parallel to the base. $\mathcal{S}$ divides $\mathcal{C}$ into three solid regions: the interior of $\mathcal{S}$, the region "below" $\mathcal{S}$ and "above" the base of $\mathcal{C}$, and the region $\mathcal{R}$ "above" $\mathcal{S}$. Compute the volume of $\mathcal{R}$.

7. **Let's get to the bottom of this.** Let $f(x,y) = x^2 + y^2 - 4\sin(xy)$.

   (a) Assuming that $f(x,y)$ actually attains a minimum value as (x,y) ranges over the plane, find this minimum value.

   (b) Prove that $f(x,y)$ actually *does* attain a minimum value (which was to be computed in part (a)).

8. **The big race.** Laura is competing in the 100-yard dash. For at least the first 60 yards, her (varying) speed in yards per second is proportional to the cube root of the distance she has run in yards. When she has run 16 yards, her speed is 6 yards per second. How many seconds does it take Laura to run the first 54 yards? Justify your answer.

Now wasn't that fun?