# SOLUTIONS TO 2016 UCONN UNDERGRADUATE CALCULUS COMPETITION EXAM

Tuesday 22 March 2016, 6:30-8:00 p.m.

Please show enough of your work so your line of reasoning will be clear. Numerical answers will receive no credit if they are not adequately supported. Calculators are welcome, but unlikely to be very useful. Have fun, and good luck!

1. **Double tangency.** Find all quadratic polynomials $p(x) = ax^2 + bx + c$ for which the graphs of $p$ and $p'$ are tangent to one another at the point $(2, 1)$.

   **Solution.** $p'(x) = 2ax + b$ and $p''(x) = 2a$. We are given that $p(2) = p'(2) = 1$ and $p'(2) = (p')'(2) = p''(2)$ so $p''(2) = 1$. Thus $1 = 4a+2b+c = 4a+b = 2a$ . Successively, $2a = 1$ so $a = 1/2$; $2 + b = 4a + b = 1$ so $b = -1$; and $2 - 2 + c = 4a + 2b + c = 1$ so $c = 1$ . Thus the only such polynomial is $p(x) = (1/2)x^2 - x + 1$ .

2. **Box in a cone.** A right circular cone has height 9 inches and base radius 3 inches. It is desired to "inscribe" in it a rectangular box, one pair of opposite faces being squares. If one of the square faces lies on the circular base of the cone and the four vertices of the opposite square face all lie on the surface of the cone, what is the maximum possible volume of the box, and what are the dimensions of the box for which that maximum is achieved?

   **Solution.** Let the height and square side of such an inscribed box be $h$ inches and $s$ inches, so its volume is $V = s^2h$ in$^3$. At height $h$ inches from the circular base of the cone the radius $r$ inches of the circular cross-section is given by $\dfrac{r}{9-h} = \dfrac{3}{9}$, so $h = 9-3r$. Also, $s = r\sqrt{2}$. Thus $V = f(r) = (r\sqrt{2})^2(9-3r) = 2r^2(9-3r) = 6(3r^2-r^3)$. $f'(r) = 6(6r - 3r^2) = 18r(2 - r)$, so $f'(r) = 0$ if $r = 0$ or $2$. $f(0) = f(3) = 0$, so the only possible maximum for $f$ is $f(2) = 24$, when $r = 2$ and $h = 3$, hence the maximum possible volume is 24 in$^3$, attained when the box is 3 inches high and the sides of its square faces are $2\sqrt{2}$ inches long.

3. **Where have all the tangents gone?** Consider the curve $\mathcal{C}$ whose equation is $20x^2 - 12xy + y^2 + 3 = 0$ . Find all points $P(x, y)$ on $\mathcal{C}$ such that the tangent line to $\mathcal{C}$ at $P$ passes through the point $Q(0, 3)$.

   **Solution.** Differentiate the given equation with respect to $x$, thinking of $y$ as a function of $x$ with derivative $y'$: $40x - 12(y + xy') + 2yy' = 0$ so $y' = \dfrac{20x - 6y}{6x - y}$ along $\mathcal{C}$. At $P$ this derivative must equal the slope from $P$ to $Q$, that is, $\dfrac{20x - 6y}{6x - y} = \dfrac{y - 3}{x}$. Clearing denominators and simplifying gives $20x^2-12xy+y^2+18x-3y = 0$. Comparing with the equation of $\mathcal{C}$, we see that $18x - 3y = 3$ or $y = 6x - 1$. Plugging this into the equation of $\mathcal{C}$, we get $20x^2 - 12x(6x - 1) + (6x - 1)^2 + 3 = 0$ which simplifies to $-16x^2 + 4 = 0$, so $x = \pm 1/2$. Thus there are two such points $P$: $(1/2, 2)$ and $(-1/2, -4)$.

4. **Sums and logarithms.** For $n = 1, 2, \dots$ let

$$a_n = \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{3n} = \sum_{k=n+1}^{3n} \frac{1}{k} \ .$$

Thus, for instance, $a_1 = \frac{1}{2} + \frac{1}{3} = \frac{5}{6}$ and $a_2 = \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} = \frac{19}{20}$ .
Compute $\lim_{n\to\infty} a_n$, or prove that this limit does not exist.

**Solution.** Note that

$$a_n = \sum_{j=1}^{2n} \frac{1}{n+j} = \sum_{j=1}^{2n} \frac{1}{1+\frac{j}{n}} \cdot \frac{1}{n}$$

is the $(2n)^{th}$ Riemann sum for the integral $J = \int_0^2 \frac{1}{1+x}\,dx$, so

$$\lim_{n\to\infty} a_n = J = \ln(1+x)|_0^2 = \ln 3 - \ln 1 = \ln 3 .$$

5. **An integral with lots of ingredients.** Evaluate the definite integral

$$I = \int_{\ln\left(\frac{\sqrt{3}-1}{2}\right)}^{0} \frac{1}{e^x + 1 + e^{-x}}\,dx \ .$$

**Solution.** Multiply the integrand by $e^x/e^x$, substitute $u = e^x$, manipulate, then substitute $v = (2u+1)/\sqrt{3}$:

$$I = \int_{\ln\left(\frac{\sqrt{3}-1}{2}\right)}^{0} \frac{e^x}{(e^x)^2 + e^x + 1}\,dx = \int_{\frac{\sqrt{3}-1}{2}}^{1} \frac{1}{u^2+u+1}\,du =$$

$$= \int_{\frac{\sqrt{3}-1}{2}}^{1} \frac{1}{\left(u+\frac{1}{2}\right)^2 + \frac{3}{4}}\,du = \frac{4}{3}\int_{\frac{\sqrt{3}-1}{2}}^{1} \frac{1}{\frac{4}{3}\left(u+\frac{1}{2}\right)^2 + 1}\,du =$$

$$= \frac{4}{3}\int_{\frac{\sqrt{3}-1}{2}}^{1} \frac{1}{\left(\frac{2u+1}{\sqrt{3}}\right)^2 + 1}\,du = \frac{4}{3}\int_{1}^{\sqrt{3}} \frac{1}{v^2+1} \cdot \frac{\sqrt{3}}{2}\,dv =$$

$$= \frac{2}{\sqrt{3}} \arctan v|_1^{\sqrt{3}} = \frac{2}{\sqrt{3}}\left(\frac{\pi}{3} - \frac{\pi}{4}\right) = \frac{\pi\sqrt{3}}{18} .$$

6. **Max-min.** Find all absolute (or global) and relative (or local) maximum and minimum values assumed by the function $f(x, y) = x^4 - 4xy + y^2$, and identify the points at which they occur.

**Solution.** The various first-order and second-order partial derivatives of $f$ are $f_x = 4x^3 - 4y$, $f_y = -4x + 2y$, $f_{xx} = 12x^2$, $f_{xy} = f_{yx} = -4$, $f_{yy} = 2$. The critical points occur when $f_x = f_y = 0$, that is, when $y = 2x = x^3$. Solving for $x$, $x(2 - x^2) = 0$, so $x = 0, \pm\sqrt{2}$ and the critical points are $P_0 = (0, 0), P_+ = (\sqrt{2}, 2\sqrt{2})$, and $P_- = (-\sqrt{2}, -2\sqrt{2})$. The determinant of the Hessian matrix of second-order partial derivatives of $f$ is $D = f_{xx}f_{yy} - f_{xy}f_{yx} = 24x^2 - 16$. Thus $D(P_0) = -16 < 0$, so $f$ has a saddle point, and no extreme value, at $P_0$. On the other hand, $D(P_+) = D(P_-) = 32 > 0$ and at both points $f_{xx} = 24 > 0$, so $f$ has at least a relative minimum value of $f(P_+) = f(P_-) = -4$ at $P_+$ and $P_-$.

Are these relative minima actually absolute minima? Note that $f(x, y) = (x^4 - 4x^2) + (2x - y)^2$. It is easy to check that $g(x) = x^4 - 4x^2$ has the minimum value -4 when $x = \pm\sqrt{2}$, so always $f(x, y) = g(x) + (2x - y)^2 \geq -4 + 0 = -4$. Thus the only extreme value of $f$ is an absolute (and relative) minimum value of -4, taken at $P_+$ and $P_-$.

7. **Fubini number formula** (Thanks to Michael Joseph for this interesting problem.) For a nonnegative integer $n$, the $n^{th}$ *Fubini number* $F_n$ is the number of possible orders in which a race with $n$ participants can finish (where ties are allowed). For example, $F_2 = 3$ because a race between $A$ and $B$ might result in $A$ beating $B$, in $B$ beating $A$, or a tie. Another example: $F_3 = 13$ because the possible orders of finish in a race between $A$, $B$, and $C$ are $ABC, ACB, BAC, BCA, CAB, CBA, (AB)C, (AC)B, (BC)A, A(BC), B(AC), C(AB), (ABC)$ where parenthesis around several contestants indicate a tie. The sequence of these numbers begins $F_0 = 1, F_1 = 1, F_2 = 3, F_3 = 13, F_4 = 75, F_5 = 541$.

   It is a known fact (which you do not have to prove) that $F_n$ arises in the Taylor series expansion
$$\frac{1}{2-e^x} = \sum_{n=0}^{\infty} \frac{F_n}{n!} x^n$$
   which converges and is valid for $-\ln 2 < x < \ln 2$. Use this fact to develop a formula for $F_n$ as an infinite sum.

   **Solution.** For $-\ln 2 < x < \ln 2$ we have $0 < e^x/2 < 1$, so the geometric series expansion
$$\frac{1}{2-e^x} = \frac{1}{2} \cdot \frac{1}{1-e^x/2} = \frac{1}{2} \sum_{k=0}^{\infty} \left(\frac{e^x}{2}\right)^k = \sum_{k=0}^{\infty} \frac{e^{kx}}{2^{k+1}}$$
   is valid. The series for the exponential function can be inserted and the order of summation reversed; this last is legitimate because, at least for $0 < x < \ln 2$, all terms are positive. Thus
$$\frac{1}{2-e^x} = \sum_{k=0}^{\infty} \frac{1}{2^{k+1}} \sum_{n=0}^{\infty} \frac{(kx)^n}{n!} = \sum_{n=0}^{\infty} \frac{1}{n!} \left( \sum_{k=0}^{\infty} \frac{k^n}{2^{k+1}} \right) x^n$$
   (with the usual convention $0^0 = 1$). Comparing the coefficient of $x^n$ in this equation with the equation given in the statement of the problem, we see that
$$F_n = \sum_{k=0}^{\infty} \frac{k^n}{2^{k+1}}.$$
   For $n \geq 1$ the series can begin with $k = 1$.

8. **Going off on a tangent.** Suppose that $y = f(x)$ is a solution to the initial value problem $\dfrac{dy}{dx} = x^2 + y^2 + 1$ , $y(0) = 0$ , valid on the interval $(-\delta, \delta)$ for some positive number $\delta$. Show that $f(x) > \tan x$ for $0 < x < \delta$ .

   *Comment:* So necessarily $\delta \leq \frac{\pi}{2}$ . In fact, $\delta < \frac{\pi}{2}$ .

   **Solution.** If $0 < x < \delta$, $f'(t) > f(t)^2 + 1$ for $0 < t < x$, so
$$x = \int_0^x 1\,dt < \int_0^x \frac{f'(t)}{f(t)^2+1}\,dt = \arctan f(t)|_0^x = \arctan f(x).$$
   Applying the tangent to both sides gives $\tan x < f(x)$ .