2016 UCONN UNDERGRADUATE CALCULUS COMPETITION

Tuesday 22 March 2016, 6:30-8:00 p.m.

Please show enough of your work so your line of reasoning will be clear. Numerical answers will receive no credit if they are not adequately supported. Calculators are welcome, but unlikely to be very useful. Have fun, and good luck!

1. **Double tangency.** Find all quadratic polynomials $p(x) = ax^2 + bx + c$ for which the graphs of p and p' are tangent to one another at the point $(2, 1)$.

2. **Box in a cone.** A right circular cone has height 9 inches and base radius 3 inches. It is desired to "inscribe" in it a rectangular box, one pair of opposite faces being squares. If one of the square faces lies on the circular base of the cone and the four vertices of the opposite square face all lie on the surface of the cone, what is the maximum possible volume of the box, and what are the dimensions of the box for which that maximum is achieved?

3. **Where have all the tangents gone?** Consider the curve $\mathcal{C}$ whose equation is $20x^2 - 12xy + y^2 + 3 = 0$. Find all points $P(x, y)$ on $\mathcal{C}$ such that the tangent line to $\mathcal{C}$ at P passes through the point $Q(0, 3)$.

4. **Sums and logarithms.** For $n = 1, 2, \ldots$ let
$$a_n = \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{3n} = \sum_{k=n+1}^{3n} \frac{1}{k} \ .$$
Thus, for instance, $a_1 = \frac{1}{2} + \frac{1}{3} = \frac{5}{6}$ and $a_2 = \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} = \frac{19}{20}$.
Compute $\lim_{n\to\infty} a_n$, or prove that this limit does not exist.

5. **An integral with lots of ingredients.** Evaluate the definite integral
$$I = \int_{\ln\left(\frac{\sqrt{3}-1}{2}\right)}^{0} \frac{1}{e^x + 1 + e^{-x}}\, dx \ .$$

6. **Max-min.** Find all absolute (or global) and relative (or local) maximum and minimum values assumed by the function $f(x, y) = x^4 - 4xy + y^2$, and identify the points at which they occur.

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7. **Fubini number formula** (Thanks to Michael Joseph for this interesting problem.) For a nonnegative integer n, the n^{th} *Fubini number* F_n is the number of possible orders in which a race with n participants can finish (where ties are allowed). For example, $F_2 = 3$ because a race between A and B might result in A beating B, in B beating A, or a tie. Another example: $F_3 = 13$ because the possible orders of finish in a race between A, B, and C are $ABC, ACB, BAC, BCA, CAB, CBA, (AB)C, (AC)B, (BC)A, A(BC)$, $B(AC), C(AB), (ABC)$ where parenthesis around several contestants indicate a tie. The sequence of these numbers begins $F_0 = 1, F_1 = 1, F_2 = 3, F_3 = 13, F_4 = 75, F_5 = 541$.

   It is a known fact (which you do not have to prove) that F_n arises in the Taylor series expansion
$$\frac{1}{2 - e^x} = \sum_{n=0}^{\infty} \frac{F_n}{n!} x^n$$
   which converges and is valid for $-\ln 2 < x < \ln 2$. Use this fact to develop a formula for F_n as an infinite sum.

8. **Going off on a tangent.** Suppose that $y = f(x)$ is a solution to the initial value problem $\dfrac{dy}{dx} = x^2 + y^2 + 1$, $y(0) = 0$, valid on the interval $(-\delta, \delta)$ for some positive number δ. Show that $f(x) > \tan x$ for $0 < x < \delta$.

   Comment: So necessarily $\delta \leq \frac{\pi}{2}$. In fact, $\delta < \frac{\pi}{2}$.

Now wasn't that fun?