SOLUTIONS TO 2015 UCONN UNDERGRADUATE CALCULUS COMPETITION

Tuesday 24 March 2015, 7:00-8:30 p.m.

Please show enough of your work so your line of reasoning will be clear. Numerical answers will receive no credit if they are not adequately supported. Calculators are welcome, but unlikely to be very useful. Have fun, and good luck!

1. **Peaks and valleys.** Find the maximum and minimum values of the function $f(x) = (x^2-4)^8 - 128\sqrt{4-x^2}$ over its domain.

   Solution. For x to be in the domain of f, the expression inside the radical must be nonnegative, so the domain is $[-2,2]$. Now we let $u = \sqrt{4-x^2}$ for $-2 \le x \le 2$, so $0 \le u \le 2$, and substitute:

   $$f(x) = (-u^2)^8 - 128u = u^{16} - 128u = g(u), 0 \le u \le 2.$$

   $g'(u) = 16u^{15} - 128 = 0$ if $u^{15} = 8, u = 8^{1/15} = 2^{1/5}$. $g(0) = 0$, $g(2) = 2^{16} - 128 \cdot 2 = 2^8\left(2^8-1\right) = 65,280$, and $g\left(2^{1/5}\right) = 2^{16/5} - 128 \cdot 2^{1/5} = \left(2^3 - 128\right) \cdot 2^{1/5} = -120 \cdot 2^{1/5}$. Thus $\max f = \max g = 65,280$ and $\min f = \min g = -120 \cdot 2^{1/5} \sim -137.8438$.

2. **An area.** Find the total area of the bounded plane region(s) enclosed by the curves $y = \frac{1}{2}x - \frac{1}{2}x^{2/3}$ and $x = y^3$.

   Solution. The curves meet when $y = \frac{1}{2}y^3 - \frac{1}{2}\left(y^3\right)^{2/3} = \frac{1}{2}\left(y^3 - y^2\right)$, or $0 = y^3 - y^2 - 2y = y(y+1)(y-2)$, so $y = 0, 1$ or 2. Thus $(x,y) = (-1,-1), (0,0)$, or $(8,2)$. $x = y^3 \iff y = x^{1/3}$. Let $\Delta(x) = \left(\frac{1}{2}\left(x^{1/3}\right)^3 - \frac{1}{2}\left(x^{1/3}\right)^2\right) - x^{1/3} =$ $= \frac{1}{2}x^{1/3}\left(x^{1/3}+1\right)\left(x^{1/3}-2\right)$. $-1 < x < 0 \Longrightarrow \Delta(x) > 0$, and $0 < x < 8 \Longrightarrow \Delta(x) < 0$, so

   $$\text{area} = \int_{-1}^{0} \Delta(x)\,dx + \int_{0}^{8} (-\Delta(x))\,dx =$$

   $$= \left[\frac{1}{4}x^2 - \frac{3}{10}x^{5/3} - \frac{3}{4}x^{4/3}\right]_{-1}^{0} + \left[-\frac{1}{4}x^2 + \frac{3}{10}x^{5/3} + \frac{3}{4}x^{4/3}\right]_{0}^{8} =$$

   $$= \left(0 - \left[\frac{1}{4} + \frac{3}{10} - \frac{3}{4}\right]\right) + \left(\left[-16 + \frac{48}{5} + 12\right] - 0\right) = \frac{1}{5} + \frac{28}{5} = \frac{29}{5}\ (= 5.8)\,.$$

3. **The lost constant.** The point $(2,1)$ is on the curve $x^4 + ky^4 = 16 + k$ no matter what the constant k is. For one particular nonzero choice of k, $y'(2) = y''(2)$ along this curve. Find the value of this special choice for k.

   Solution. For this k, let $t = y'(2) = y''(2)$. Differentiate implicitly twice:

   $$4x^3 + 4ky^3y' = 0 = 4\left(3x^2\right) + 4k\left(3y^2y' \cdot y' + y^3 \cdot y''\right).$$

   Plug in $x = 2, y = 1$: $32 + 4kt = 0 = 48 + 4k\left(3t^2 + t\right) = 48 + 4kt(3t+1)$. The first equation gives $4kt = -32$ and $kt = -8$, so the second now gives $0 = 48 - 32(3t+1)$ or $3t + 1 = 3/2$ and $t = 1/6$. Finally, we get

   $$k = \frac{-8}{t} = \frac{-8}{1/6} = -48.$$

4. **The biggest cylinder.** A right circular cone has height 9 and a circular base of radius 6. Find the largest possible volume of a right circular cylinder inscribed in the cone with one end on the base of the cone.

   Solution. Place the entire figure in xyz-space so that the circular base of the cone is in the xy-plane with center $(0,0,0)$ and radius 6, while the apex is at $(0,0,9)$. The cylinder has radius r $(0 \le r \le 6)$, and its top is at height h $(0 \le h \le 9)$ where $9r + 6h = 54$, so $h = \dfrac{54 - 9r}{6} = 9 - \dfrac{3}{2}r$. As a function of r, the volume of the cylinder is $V = \pi r^2 h = \pi r^2(9 - (3/2)r) = (\pi/2)(18r^2 - 3r^3) = f(r)$. $f'(r) = (\pi/2)(36r - 9r^2) = (9\pi/2)r(4 - r)$ which, for $0 < r < 6$, equals 0 only when $r = 4$. $f(0) = f(6) = 0$ and $f(4) = 48\pi$, which is the largest possible volume.

5. **How cool is cool?** According to Newton's law of cooling, the rate at which a cup of coffee cools is proportional to the difference between its temperature and that of the room it is in. A certain cup of coffee cools from 164° to 140° (all temperatures Fahrenheit) in five minutes, and then from 140° to 122° in the next five minutes. What is the temperature of the room?

   Solution. Let t be the time in minutes after cooling begins, and let $f(t)^\circ$ be the temperature of the coffee at time t. Let c° be the temperature of the room. Newton's law says that $f'(t) = k(f(t) - c)$ for some constant k. If $g(t) = f(t) - c$, then $g'(t) = kg(t)$, so $g(t) = \lambda e^{kt}$ for yet another constant λ. Thus $f(t) = g(t) + c = \lambda e^{kt} + c$. We are given that $164 = f(0) = \lambda + c$, so $\lambda = 164 - c$; that $140 = f(5) = \lambda e^{5k} + c =$ $= (164 - c)u + c$ where $u = e^{5k}$; and that $122 = f(10) = \lambda e^{10k} + c = (164 - c)u^2 + c$. Thus $u = \dfrac{140 - c}{164 - c}$ and $u^2 = \dfrac{122 - c}{164 - c}$. But also $u^2 = \left(\dfrac{140 - c}{164 - c}\right)^2$. Equating the two expressions for u^2 and multiplying by $(164 - c)^2$ gives $(140 - c)^2 = (122 - c)(164 - c)$ or $c^2 - 280c + 19{,}600 = c^2 - 286c + 20{,}008$, so $6c = 408$ and $c = 68$. The temperature of the room is 68°.

6. **A tricky trig integral.** Evaluate the integral

$$I = \int_{\pi/4}^{\pi/3} \frac{1}{\tan\theta + \cot\theta}\, d\theta.$$

Solution. $I = \displaystyle\int_{\pi/4}^{\pi/3} \frac{1}{\tan\theta + \cot\theta} \cdot \frac{\sin\theta\cos\theta}{\sin\theta\cos\theta}\, d\theta = \int_{\pi/4}^{\pi/3} \frac{\sin\theta\cos\theta}{\sin^2\theta + \cos^2\theta}\, d\theta =$
$= \displaystyle\int_{\pi/4}^{\pi/3} \sin\theta\cos\theta\, d\theta$. Substituting $u = \sin\theta, du = \cos\theta\, d\theta$ gives
$I = \displaystyle\int_{\sqrt{2}/2}^{\sqrt{3}/2} u\, du = \frac{u^2}{2}\Big|_{\sqrt{2}/2}^{\sqrt{3}/2} = \frac{3/4}{2} - \frac{2/4}{2} = 1/8.$

7. **A trig series.** Determine (proof really needed!) whether the infinite series

$$\sum_{n=1}^{\infty} \left(1 - \cos\frac{\pi}{n}\right)$$

converges.

Solution. $1 - \cos x \geq 0$ for all x. Let $f(x) = x^2 - (1 - \cos x)$. $f'(x) = 2x - \sin x$ and $f''(x) = 2 - \cos x > 0$, so f' is strictly increasing. $f'(0) = 0$, so $f'(x) > 0$ for $x > 0$ and f is strictly increasing for $x \geq 0$. $f(0) = 0$, so $f(x) > 0$, that is, $1 - \cos x < x^2$, for all $x > 0$. Thus $0 \leq 1 - \cos\frac{\pi}{n} < \left(\frac{\pi}{n}\right)^2$. $\displaystyle\sum_{n=1}^{\infty}\left(\frac{\pi}{n}\right)^2 = \pi^2 \sum_{n=1}^{\infty} \frac{1}{n^2}$. The last sum converges (p-series), hence by the comparison test so does the given series.

8. **Cutting a cone.** A cone in xyz-space has as its cross-section at height z a circle centered at $(0, 0, z)$ of radius $|z|$. Consider the solid $\mathcal{S}$ consisting of those points which lie inside the cone, above the xy-plane, and below the planes $z = 3$ and $z = 2x - 1$. Set up, but *do not evaluate*, an integral or iterated double integral or iterated triple integral (or sum of such integrals) whose value is the volume of $\mathcal{S}$. There may be many correct answers; for whatever answer you give, the crucial things to get right are the integrand(s) and all the limits of integration.

Solution. $\mathcal{S}$ consists of those points (x, y, z) that satisfy $0 \leq z \leq 3$, $\dfrac{z+1}{2} \leq x \leq z$, and $x^2 + y^2 \leq z^2$. The second of these forces $\dfrac{z+1}{2} \leq z$, so in fact $z \geq 1$. Fixing appropriate z and x, we have $|y| \leq \sqrt{z^2 - x^2}$. This gives us one possible solution (there are others): the volume of $\mathcal{S}$ is equal to

$$\int_1^3 \left(\int_{\frac{z+1}{2}}^{z} 2\sqrt{z^2 - x^2}\, dx \right) dz.$$