2014 UCONN UNDERGRADUATE CALCULUS COMPETITION

Thursday 27 March 2014, 6:00-7:30 p.m.

Please show enough of your work so your line of reasoning will be clear. Numerical answers will receive no credit if they are not adequately supported. Calculators are welcome, but unlikely to be very useful. Have fun, and good luck!

1. **A parabolic tangency problem.** There is exactly one positive number a such that the parabola whose equation is $y = 9 + x^2$ is tangent to each of the lines $y = ax$ and $y = -ax$. Find a and the points of tangency, and calculate the area of the plane region bounded by the parabola $y = 9 + x^2$ and the lines $y = ax$ and $y = -ax$.

2. **A box in a cone.** A right circular cone has height 2 and a base of radius 1. A rectangular box with square top and bottom is to be inscribed in it in such a way that the square bottom is on the circular base of the cone and the four corners of the square top lie on the surface of the cone. Find the maximum possible volume of such a box.

3. **An inverse integral.** The function $f(x) = x^3 + x$ is strictly increasing, and so has an inverse function $g = f^{-1}$, that is, $g(f(x)) = x = f(g(x))$ for all real numbers x. Evaluate $\displaystyle\int_2^{10} g(x)\,dx$.

4. **¡Viva la revolución!** For a positive constant k, consider the region $\mathcal{R}$ bounded by the line $y = kx$ and the parabola $y = x^2$. There is only one choice of k such that the solids obtained by revolving $\mathcal{R}$ about the x-axis and the y-axis have the same volume. Find the value of this special choice of k

5. **The bottom of the basin.** Assuming that the function $f(x, y) = x^2 + y^2 - 4\sin(xy)$ actually attains an absolute minimum value as (x, y) ranges over the plane, find this minimum value.

6. **Convergence of a sequence.** Let $a_1, a_2, a_3, \ldots, a_n, \ldots$ be the sequence defined by

$$a_n = 2\sqrt{n} - \sum_{k=1}^{n} \frac{1}{\sqrt{k}} = 2\sqrt{n} - \frac{1}{\sqrt{1}} - \frac{1}{\sqrt{2}} - \cdots - \frac{1}{\sqrt{n}}\,.$$

Show that the sequence $\{a_n\}$ is convergent to some limit L, and that $1 < L < 2$.

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7. **A Fibonacci series.** (Thanks to Michael Joseph for this lovely problem.) Let F_n be the n^{th} term of the Fibonacci sequence. That is, $F_1 = F_2 = 1$ and F_n is defined recursively for $n \geq 3$ by $F_n = F_{n-2} + F_{n-1}$. It is a known fact (which you don't have to prove) that

$$\lim_{n\to\infty} \frac{F_{n+1}}{F_n} = \frac{1+\sqrt{5}}{2}.$$

Show that the series

$$\sum_{n=1}^{\infty} \frac{F_n}{2^n}$$

is convergent and compute its sum.

8. **A nonlinear differential equation.** Show that any solution $y = f(x)$ of the second order differential equation $y'' + e^{y^2} = 0$ must be bounded above on any interval I on which it is defined; that is, there is a finite constant M such that $f(x) \leq M$ for every x in I. [Two-word hint: *integrating factor*]

Now wasn't that fun?