Problem Seminar, Spring 2017-Calculus II

1. (Difficulty 2) Show that if a, b, c are positive numbers such that $a+b+c = abc$ then
$$\frac{1}{\sqrt{1+a^2}} + \frac{1}{\sqrt{1+b^2}} + \frac{1}{\sqrt{1+c^2}} \leq \frac{3}{2}.$$
Hint: Use convexity.

2. (Difficulty 3) Let $f : \mathbb{R} \to (0, +\infty)$ be a continuous function. Show that
$$\int_0^1 f(x)dx \geq 2^{\int_0^1 \log_2 f(t)dt}.$$

3. (Difficulty 4) Let $f : [0,1] \to \mathbb{R}$ be a continuous function such that $\int_0^1 f(x)dx = 1$ and
$$\int_0^1 (1 - f(x))e^{-f(x)}dx \leq 0.$$
Show that $f(x) = 1$ for all $x \in [0,1]$.

4. (Difficulty 3) Prove that for every $r > 0$ the set
$$V_{f,g} = \{(x, y, z) \in \mathbb{R}^3 : (x - f(z))^2 + (y - g(z))^2 \leq r,\ z \in [a, b]\}$$
has the same volume for all continuous functions $f, g : [a, b] \to \mathbb{R}$.

5. (Difficulty 4) Show that the integral
$$\lim_{B\to\infty} \int_0^B \sin(x)\sin(x^2)dx$$
converges.

6. (Difficulty 3) Let $f : \mathbb{R} \to (0, +\infty)$ be a continuous 1-periodic function. Prove that
$$\int_0^1 \frac{f(x)}{f(x + 1/2)}dx \geq 1.$$

7. (Difficulty) Let $\mathcal{F}$ be a finite collection of open discs in $\mathbb{R}^2$ whose union contains a set $E \subset \mathbb{R}^2$. Show that there exists a pairwise disjoint subcollection $\{D_i = B(x_i, r_i)\}_{i=1}^n$ in $\mathcal{F}$ such that
$$E \subset \cup_{i=1}^n B(x_i, 3r_i).$$

8. (Difficulty 10+) Does there exist a continuous function $f : \mathbb{R} \to \mathbb{R}$ such that $f(f(x)) = e^x$ for all $x \in \mathbb{R}$.