Putnam Calculus Problems 2016-II

1. (Difficulty 1) Prove that
$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}(n+1)} < 2.$$

2. (Difficulty 2) Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable and invertible function such that f^{-1} is differentiable as well. The graph of f intersects the line $y = x$ at the points (a, a) and (b, b) with $0 < a < b$. Show that
$$\int_a^b (f(x) + f^{-1}(x))dx = b^2 - a^2.$$

3. (Difficulty 2) The functions $f, g : \mathbb{R} \to \mathbb{R}$ are continuous and for all $x \in \mathbb{R}$, $g(x) \neq 0$ and
$$\int_1^x f(t)dt = 2 + x\int_0^x g(t)dt.$$
Show that there exists $x_0 \in (0, 1)$ such that
$$f(x_0) = 2g(x_0) + 2.$$

4. (Difficulty 3) Let $f : \mathbb{R} \to \mathbb{R}$ such that
$$f'(x) = \frac{2e^x + x^2}{e^x + x^2}$$
for all $x \in \mathbb{R}$. Show that
$$\lim_{x\to+\infty}(f(x + 2) - f(x)) = 4.$$

5. (Difficulty 4) For $n \in \mathbb{N}$ and $x \in \mathbb{R}$ let $f_n(x) = \prod_{i=1}^{2n} \sin(2^i x)$. Show that for all $n \in \mathbb{N}$ and $x \in \mathbb{R}$,
$$|f_n(x)| \leq \frac{2}{\sqrt{3}}|f_n(\pi/3)|.$$

6. (Difficulty 4) For which $c \in \mathbb{R}$,
$$\frac{1}{2}(e^x + e^{-x}) \leq e^{cx^2}$$
for all $x \in \mathbb{R}$.

7. (Difficulty 5) Prove that there is no function $f : \mathbb{R} \to \mathbb{R}$ with $f(0) > 0$ such that
$$f(x + y) \geq f(x) + yf(f(x))$$
for all $x, y \in \mathbb{R}$

8. (Difficulty 6) Let P be a non-constant polynomial with real coefficients. Prove that there exist finitely many $x > 0$ such that
$$\int_0^x P(t)\cos t dt = \int_0^x P(t)\sin t dt = 0.$$

9. (Difficulty 8) Let $(p_i)_{i\in\mathbb{N}}$ be a sequence of points in $\mathbb{R}^3$ satisfying

   (a) $|p_i| \geq 1$ for all $i \in \mathbb{N}$,

   (b) $|p_i - p_j| \geq 1$. for all $i, j \in \mathbb{N}$ with $i \neq j$.

   Prove that, if $\lambda > 3$, then the series

   $$\sum_{i=1}^{\infty} \frac{1}{|p_i|^{\lambda}}$$

   converges.

10. (Difficulty 10+) Does there exist a continuous function $f : \mathbb{R} \to \mathbb{R}$ such that $f(f(x)) = e^x$ for all $x \in \mathbb{R}$.