Putnam Calculus Problems 2016-I

1. (Difficulty 1) Show that for all $x > 1$,
$$\int_1^x e^{-t^2} dt < \frac{1}{2e}.$$

2. (Difficulty 2) Let $R > 0$. Find the limit
$$\lim_{n\to\infty} \int_0^R \sin(x^n) dx.$$

3. (Difficulty 3) Let $f(x) = \int_x^{x^2} \frac{dy}{\ln y}$ for $x \in (1, \infty)$. Show that f is injective on $(1, \infty)$ and find the range of f.

4. (Difficulty 3) Suppose that $f : \mathbb{R} \to \mathbb{R}$ satisfies,
$$\left| \sum_{j=1}^{n} 10^j (f(x + jy) - f(x - jy)) \right| \le 1$$
for all $n \in \mathbb{N}$ and $x, y \in \mathbb{R}$. Show that f is constant.

5. (Difficulty 3) If f is continuous find the limit
$$\lim_{n\to\infty} \frac{1}{n} \int_0^1 x^{1/n-1} f(x) dx.$$

6. (Difficulty 4)Let $f : (0, +\infty) \to \mathbb{R}$ be a twice differentiable function such that
$$\lim_{x\to 0^+} f'(x) = -\infty \text{ and } \lim_{x\to 0^+} f''(x) = +\infty.$$
Show that
$$\lim_{x\to 0^+} \frac{f(x)}{f'(x)} = 0.$$

7. (Difficulty 5) Let $(r_n)_{n\in\mathbb{N}}$ be a sequence of positive reals. Find the limit
$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \ln\left(\frac{k}{n} + r_n\right).$$

8. (Difficulty 6) Let f be a 3-times continuously differentiable function $f : \mathbb{R} \to \mathbb{R}$ such that
   (i) $f(0) = 0$,
   (ii) $f'(0) = 0$,
   (iii) $f''(0) > 0$.

   Let
$$g(x) = \left(\frac{\sqrt{f(x)}}{f'(x)} \right)'.$$
for $x \ne 0$ and $g(0) = 0$. Show that g is bounded in some neighborhood of 0.

9. (Difficulty 7) Find the maximum value of

$$\int_0^y \sqrt{x^4 + (y - y^2)^2} dx$$

for $0 \leq y \leq 1$.

10. (Difficulty 8) Does there exists an infinite sequence of closed discs $(D_i)_{i=1}^{\infty}$ in the plane such that

    (i) the centers of the c_i have no finite limit point,

    (ii) the sum of the areas of the D_i is finite, and

    (iii) every line in the plane intersects at least one of the D_i?