GRE Subject test preparation – Spring 2016

Topic: Abstract Algebra, Linear Algebra, Number Theory.

Linear Algebra

- Standard matrix manipulation to compute the kernel, intersection of subspaces, column spaces, and such.
- The following are equivalent for an $n \times n$-matrix A (acting on $\mathbb{R}^n$)
  - A is invertible, i.e. there exists a matrix B such that $BA = I$ (which is equivalent to exists a matrix B such that $AB = I$).
  - $\mathrm{Ker} A = 0$, i.e. no non-zero vector $v \in \mathbb{R}^n$ such that $Av = 0$.
  - A is surjective, i.e. for every $w \in \mathbb{R}^n$, there exists $v \in \mathbb{R}^n$ such that $Av = w$.
  - $\det A \neq 0$.
  - The columns/rows of A are linearly independent/span $\mathbb{R}^n$.
  - The transpose A^T is invertible.
  - 0 is not an eigenvalue of A
- Kernel/cokernel numerics. Let $f : V \to W$ be a map of finite dimensional linear spaces. Then

$$\dim V = \dim \mathrm{Ker} f + \dim(\mathrm{Im} f)$$

- Intersection numerics: if W_1 and W_2 are subspaces of a finite dimensional vector space V, then

$$\dim W_1 + \dim W_2 = \dim(W_1 \cap W_2) + \dim(W_1 + W_2).$$

- Eigenvalues, Jordan blocks, and Cayley-Hamilton theorem. Let $A \in M_{n\times n}(\mathbb{C})$. Then A is similar to a Jordan normal forms, which is a block diagonal of bunch of Jordan block:

$$J_r(\lambda) := \begin{pmatrix} \lambda & 1 & 0 & \cdots & 0 & 0 \\ 0 & \lambda & 1 & \cdots & \vdots & \vdots \\ 0 & 0 & \lambda & \cdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda & 1 \\ 0 & 0 & 0 & \cdots & 0 & \lambda \end{pmatrix}$$

  - The characteristic polynomial of A is

$$\prod_{\lambda \text{ eigenvalues}} (x - \lambda)^{\text{total size of } \lambda\text{-Jordan blocks}}$$

  - The minimal polynomial $m_A(x)$ of A (namely, the monic polynomial of minimal degree such that $m_A(A)$) is

$$\prod_{\lambda \text{ eigenvalues}} (x - \lambda)^{\text{maximal size of } \lambda\text{-Jordan blocks}}$$

  - So if A satisfies a polynomial with no multiple zeros, then A is diagonal.

  - Traces of A is the sum of all eigenvalues (with multiplicity). $\det(A)$ is the product of all eigenvalues (with multiplicity).

Abstract Algebra

- Basic examples of groups: cyclic groups, abelian groups, dihedral groups, S_n, A_n.
- All groups of prime order is cyclic.
- $\mathbb{Z}_p^\times$ is a cyclic group of order $p-1$.
- The generators of $\mathbb{Z}_n$ is all the elements that are coprime to n.

Number Theory

- Modulo arithmetic
- Prime factorization

Linear Algebra

Problem 1 If V and W are 2-dimensional subspaces of $\mathbb{R}^4$,what are the possible dimensions of the subspace $V \cap W$?

(A) 1 only (B) 2 only (C) 0 and 1 only (D) 0, 1, and 2 only

(E) 0, 1, 2, 3, and 4

Problem 2 Let A be a 2×2-matrix for which there is a constant k such that the sum of the entries in each row and each column is k. Which of the following must be an eigenvector of A?

(I) $\binom{1}{0}$ (II) $\binom{0}{1}$ (III) $\binom{1}{1}$

(A) I only (B) II only (C) III only (D) I and II only (E) I, II, and III

Problem 3 Let V be the real vector space of all real 2×3-matrices, and let W be the real vector space of all real 4×1 column vectors. If T is a linear transformation from V onto W, what is the dimension of the subspace $\{v \in V : T(v) = 0\}$?

(A) 2 (B) 3 (C) 4 (D) 5 (E) 6

Problem 4 Consider the two planes $x + 3y - 2z = 7$ and $2x + y - 3z = 0$ in $\mathbb{R}^3$. Which of the following sets is the intersection of these planes?

(A) $\emptyset$

(B) $\{(0, 3, 1)\}$

(C) $\{(x, y, z) : x = t, y = 3t, z = 7 - 2t,\ t \in \mathbb{R}\}$

(D) $\{(x, y, z) : x = 7t, y = 3 + t, z = 1 + 5t,\ t \in \mathbb{R}\}$

(E) $\{(x, y, z) : x - 2y - z = -7\}$

Problem 5 Let M be a 5×5 real matrix. Exactly four of the following five conditions on M are equivalent to each other. Which of the five conditions is equivalent to NONE of the other four?

(A) For any two distinct column vectors u and v of M, the set $\{u, v\}$ is linearly independent.

(B) The homogeneous system $Mx = 0$ has only the trivial solution.

(C) The system of equations $Mx = b$ has a unique solution for each real 5×1 column vector b.

(D) The determinant of M is nonzero.

(E) There exists a 5×5 real matrix N such that NM is the 5×5 identity matrix.

Problem 6 Suppose A and B are $n \times n$ invertible matrices, where $n > 1$, and I is the $n \times n$ identity matrix. If A and B are similar matrices, which of the following statements must be true?

I. $A - 2I$ and $B - 2I$ are similar matrices.

II. A and B have the same trace.

III. A^{-1} and B^{-1} are similar matrices.

(A) I only (B) II only (C) III only (D) I and III only (E) I, II, and III

Problem 7 The rank of the the matrix $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 6 & 7 & 8 & 9 & 10 \\ 11 & 12 & 13 & 14 & 15 \\ 16 & 17 & 18 & 19 & 20 \\ 21 & 22 & 23 & 24 & 25 \end{pmatrix}$ is

(A) 1 (B) 2 (C) 3 (D) 4 (E) 5

Problem 8 Let V be the vector space, under the usual operations, of real polynomials that are of degree at most 3. Let W be the subspace of all polynomials $p(x)$ in V such that $p(0) = p(1) = p(-1) = 0$. Then $\dim V + \dim W$ is

(A) 4 (B) 5 (C) 6 (D) 7 (E) 8

Problem 9 Let $I \neq A \neq -I$, where I is the identity matrix and A is a real 2×2 matrix. If $A = A^{-1}$, then the trace of A is

(A) 2 (B) 1 (C) 0 (D) -1 (E) -2

Problem 10 For what value (or values) of m is the vector $(1, 2, m, 5)$ a linear combination of the vectors $(0, 1, 1, 1)$, $(0, 0, 0, 1)$, and $(1, 1, 2, 0)$?

(A) For no value of m (B) -1 only (C) 1 only (D) 3 only

(E) For infinitely many values of m

Problem 11 Suppose B is a basis for a real vector space V of dimension greater than 1. Which of the following statements could be true?

(A) The zero vector of V is an element of B.

(B) B has a proper subset that spans V.

(C) B is a proper subset of a linearly independent subset of V.

(D) There is a basis for V that is disjoint from B.

(E) One of the vectors in B is a linear combination of the other vectors in B.

Problem 12 Let V and W be 4-dimensional subspace of a 7-dimensional vector space X. Which of the following CANNOT be the dimension of the subspace $V \cap W$?

(A) 0 (B) 1 (C) 2 (D) 3 (E) 4

Problem 13 For $A = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 2 & 3 & 4 & 5 \\ 0 & 0 & 3 & 4 & 5 \\ 0 & 0 & 0 & 4 & 5 \\ 0 & 0 & 0 & 0 & 5 \end{pmatrix}$, which of the following statements about A is FALSE?

(A) A is invertible

(B) If $x \in \mathbb{R}^5$ and $Ax = x$, then $x = 0$

(C) The last row of A^2 is $(0\ 0\ 0\ 0\ 25)$

(D) A can be transformed into 5×5 identity matrix by a sequence of elementary row operations

(E) $\det(A) = 120$

Problem 14 Let V be a finite-dimensional real vector space and let P be a linear transformation of V such that $P^2 = P$. Which of the following must be true?

I. P is invertible

II. P is diagonalizable

III. P is either the identity transformation or the zero transformation.

(A) None (B) I only (C) II only (D) III only (E) II and III

Problem 15 Which of the following is an <u>orthonormal</u> basis for the column space of the real matrix $\begin{pmatrix} 1 & -1 & 2 & -3 \\ -1 & 1 & -3 & 2 \\ 2 & -2 & 5 & -5 \end{pmatrix}$?

(A) $\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}$

(B) $\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$

(C) $\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \\ 0 \end{pmatrix} \right\}$

(D) $\left\{ \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ -3 \\ 5 \end{pmatrix} \right\}$

(E) $\left\{ \begin{pmatrix} \frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{6}} \\ \frac{2}{\sqrt{6}} \end{pmatrix}, \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{pmatrix} \right\}$

Problem 16 Let A be a real 3×3. Which of the following conditions does NOT imply that A is invertible?

(A) $-A$ is invertible

(B) There exists a positive integer k such that $\det(A^k) \neq 0$

(C) There exists a positive integer such that $(I - A)^k = 0$, where I is the 3×3 identity matrix

(D) The set of all vectors of the form Av, where $v \in \mathbb{R}^3$, is $\mathbb{R}^3$

(E) There exists 3 linearly independent vectors $v_1, v_2, v_3 \in \mathbb{R}^3$ such that $Av_i \neq 0$ for each i.

Problem 17 Let A be a 2×2 matrix with characteristic polynomial $p(x) = x^2 + 2x + 1$, which of the following must be true:

(I) A has an eigenvalue
(II) If A is diagonalizable, then A is the identity matrix
(III) A is invertible

(a) I only (b) II only (c) III only (d) I and III only (e) I, II, and III.

Abstract Algebra

Problem 1 For which of the following rings is it possible for the product of two nonzero elements to be zero?

(A) The ring of complex numbers

(B) The ring of integers modulo 11

(C) The ring of continuous real-valued functions on $[0, 1]$

(D) The ring $\{a + b\sqrt{2} : a \text{ and } b \text{ are rational numbers}\}$

(E) The ring of polynomials in x with real coefficients

Problem 2 Let G be the group of complex numbers $\{1, i, -1, -i\}$ under multiplication. Which of the following statements are true about the homomorphisms of G into itself?

I. $z \mapsto \bar{z}$ defines one such homomorphism, where $\bar{z}$ denotes the complex conjugate of z.

II. $z \mapsto z^2$ defines one such homomorphism.

III. For every such homomorphism, there is an integer k such that the homomorphism has the form $z \mapsto z^k$.

(A) None (B) II only (C) I and II only (D) II and III only (E) I, II, and III

Problem 3 Up to isomorphism, how many additive abelian groups G of order 16 have the property that $x + x + x + x = 0$ for each x in G?

(A) 0 (B) 1 (C) 2 (D) 3 (E) 5

Problem 4 The group of symmetries of the regular pentagram is isomorphic to the

(A) symmetric group S_5

(B) alternating group A_5

(C) cyclic group of order 5

(D) cyclic group of order 10

(E) dihedral group of order 10

Problem 5 A group G in which $(ab)^2 = a^2b^2$ for all a, b in G is necessarily

(A) finite (B) cyclic (C) of order two (D) abelian (E) none of above

Problem 6 If the finite group G contains a subgroup of order seven but no element (other than the identity) is its own inverse, then the order of G could be

(A) 27 (B) 28 (C) 35 (D) 37 (E) 42

Problem 7 Which of the following is NOT a group?

(A) The integers under addition

(B) The nonzero integers under multiplication

(C) The nonzero real numbers under multiplication

(D) The complex numbers under addition

(E) The nonzero complex numbers under multiplication

Problem 8 For which integer n such that $3 \leq n \leq 11$ is there only one group of order n (up to isomorphism)?

(A) For no such integer n

(B) For 3, 5, 7, and 11 only

(C) For 3, 5, 7, 9, and 11 only

(D) For 4, 6, 8, and 10 only

(E) For all such integers n

Problem 9 What is the largest order of an element in the group of permutations of 5 objects?

(A) 5 (B) 6 (C) 12 (D) 15 (E) 120

Problem 10 Let R be a ring and let U and V be (two-sided) ideals of R. Which of the following must also be ideals of R?

I. $U + V = \{u + v : u \in U \text{ and } v \in V\}$

II. $U \cdot V = \{uv : u \in U \text{ and } v \in V\}$

III. $U \cap V$

(A) II only (B) III only (C) I and II only (D) I and III only (E) I, II, and III

Problem 11 Let $\mathbb{Z}_{17}$ be the ring of integers modulo 17, and let $\mathbb{Z}_{17}^{\times}$ be the group of units of $\mathbb{Z}_{17}$ under multiplication. Which of the following are generators of $\mathbb{Z}_{17}^{\times}$?

I. 5 II. 8 III. 16

(A) None (B) I only (C) II only (D) III only (E) I, II, and III

Problem 12 Let G_n denote the cyclic group of order n. Which of the following are not cyclic?

(A) $G_{11} \times G_7$

(B) $G_{24} \times G_{17}$

(C) $G_{11} \times G_3 \times G_8$

(D) $G_{11} \times G_{33}$

(E) $G_{49} \times G_{121}$

Problem 13 How many generators does the group $\mathbb{Z}_{24}$ have?

(A) 2 (B) 6 (C) 8 (D) 10 (E) 12

Number Theory

Problem 1 Let P_1 be the set of all primes, $2, 3, 5, 7, \ldots$, and for each integer n, let P_n be the set of all prime multiples of $n, 2n, 3n, 5n, 7n, \ldots$. Which of the following intersections is nonempty?

(A) $P_1 \cap P_{23}$ (B) $P_7 \cap P_{21}$ (C) $P_{12} \cap P_{20}$ (D) $P_{20} \cap P_{24}$ (E) $P_5 \cap P_{25}$

Problem 2 What is the units digit in the standard decimal expansion of the number 7^{25}?

(A) 1 (B) 3 (C) 5 (D) 7 (E) 9

Problem 3 For how many positive integers k does the ordinary decimal representation of the integer $k!$ end in exactly 99 zeros?

(A) None (B) One (C) Four (D) Five (E) Twenty-four

Problem 4 Let x and y be positive integers such that $3x + 7y$ is divisible by 11. Which of the following must also be divisible by 11?

(A) $4x + 6y$ (B) $x + y + 5$ (C) $9x + 4y$ (D) $4x - 9y$ (E) $x + y - 1$

Problem 5 Which of the following CANNOT be a root of a polynomial in x of the form $9x^5 + ax^3 + b$, where a and b are integers?

(A) -9 (B) -5 (C) $\frac{1}{4}$ (D) $\frac{1}{3}$ (E) 9

Problem 6 If x and y are integers that satisfies

$$3x \equiv 5 \pmod{11} \qquad 2y \equiv 7 \pmod{11}$$

then $x + y$ is congruent modulo 11 to which of the following?

(A) 1 (B) 3 (C) 5 (D) 7 (E) 9