GRE prep questions in Analysis

1. $\int_0^{\frac{\pi}{2}} \frac{1}{1+\tan x} dx = ?$
   (a) $\pi/2$
   (b) $\pi/3$
   (c) $\pi/4$
   (d) $\pi^2/2$
   (e) $\pi^2/4$

2. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function such that $f(x+1) = f(x)$ for all $x \in \mathbb{R}$. Which of the following hold?

   (i) f achieves its maximum and minimum,

   (ii) f is uniformly continuous,

   (iii) there exists x_0 such that $f(x_0 + \pi) = f(x_0)$

   (iv) there exists x_1 such that $f(x_1 + 1/2) = f(x_1)$

   (a) (i)

   (b) (i)+(ii)

   (c) (i)+(ii)+(iv)

   (d) all

3. Let $f : \mathbb{R} \to \mathbb{R}$ differentiable. Then

$$\lim_{\varepsilon \to 0} \frac{3}{\varepsilon^3} \int_{-\varepsilon}^{\varepsilon} t f(x+t) dt =$$

(a) $\frac{1}{3} f(x)$,
(b) $\frac{1}{2} f'(x)$,
(c) $\frac{1}{2} f''(x)$,
(d) $2f'(x)$,
(e) $3f''(x)$.

4. Let (X, d) be an arbitrary metric space, which of the following define a metric on X?

   (a) $4 + d$
   (b) $e^d - 1$
   (c) $d - |d|$
   (d) d^2
   (e) $\sqrt{d}$

5. How many linearly independent solutions does the following equation have

$$e^{-x^2}y'' + \ln(1+x^2)y' - \cos(x)y = 0.$$

(a) 0

(b) 1

(c) 2

(d) undecidable

(e) infinitely many

6. Evaluate

$$\int_{-\pi/4}^{\pi/4} (\cos t + \sqrt{1+t^2}\sin^3 t \cos^3 t)dt$$

(a) 0

(b) $\sqrt{2}$

(c) $\sqrt{2}-1$

(d) $\sqrt{22}$

(e) $\frac{\sqrt{2}-1}{2}$

7. Decide which of the following hold:

(i) $\ln x \leq C\sqrt{x}$ for some constant C and $x \geq 1$,

(ii) $\sum_{k=1}^{n} k^2 \leq Cn^2$ for some constant C and $n \geq 1$,

(iii) $|\sin x - x| \leq C|x^3|$ for some constant C.

(a) None

(b) (i)+(iii)

(c) (iii)

(d) (i)

(e) all

8. Evaluate

$$\int_0^\infty \lfloor x \rfloor e^{-x} dx.$$

(a) $\frac{e}{e^2-1}$

(b) $\frac{1}{e-1}$

(c) $\frac{e-1}{e}$

(d) 1

(e) $+\infty$

9. Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function such that both limits,

$$\lim_{x\to\infty} f(x), \lim_{x\to\infty} f'(x)$$

exist and they are finite. Which of the following is true?

(a) $\lim_{x\to\infty} f(x) = 0$

(b) $\lim_{x\to\infty} f'(x) = 0$

(c) $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} f'(x)$

(d) f is constant

(e) f' is constant.

10. Let f be a continuous function on $\mathbb{R}$. Which of the following hold?

(i) $f((0,1))$ is connected.

(ii) $f((0,1))$ is open.

(iii) $f((0,1))$ is bounded.

(a) (i)

(b) all of them

(c) (i)+(ii)

(d) (i)+(iii)

(e) (ii)+(iii)

11. Suppose that f is a nonnegative differentiable function such that

$$\int_0^\infty f(x)dx < \infty.$$

Which of the following statements hold?

(i) $\lim_{x\to\infty} f(x) = 0$

(ii) $\lim_{x\to\infty} f'(x) = 0$

(iii) For every $\varepsilon > 0$, there is an M such that $\int_M^\infty f(x)dx < \varepsilon$.

(iv) f is bounded

(a) (i)

(b) (ii)+(iv)

(c) (iii)

(d) (ii)+(iii)+(iv)

(e) all of them