MATH 3974 PROBLEM SEMINAR HOMEWORK 6

1(1 pt). Prove that

$$1 - \frac{1}{x} \leq \ln x \leq x - 1,$$

for all $x \geq 1$.

2 (1 pt). Prove that

$$e^x \geq 1 + x^2,$$

for all $x \geq 0$.

3 (2 pts). Let a, b, c be the side lengths of a triangle with the property that for any positive integer n, the numbers a^n, b^n, c^n can also be the side lengths of a triangle. Prove that the triangle is necessarily isosceles.

4 (3 pts). Prove that the positive real numbers a, b, c are the side lengths of a triangle if and only if

$$a^2 + b^2 + c^2 < 2\sqrt{a^2b^2 + b^2c^2 + c^2a^2}.$$

5 (2 pts). Prove that if $z \in \mathbb{C}$ satisfies $\text{Re}z < \frac{1}{2}$, then

$$\left| \frac{z}{1 - z} \right| < 1.$$

6 (3 pts). Prove that

$$\left| \frac{z - w}{1 - \bar{z}w} \right| < 1$$

for any complex numbers z and w so that $|z| < 1$ and $|w| < 1$.

7 (3 pts). Let $z_1, .., z_n \in \mathbb{C}$ be so that $|z_1| = |z_2| = ... = |z_n| = 1$. Prove that the number

$$\omega = (z_1 + z_2 + ... + z_n) \left(\frac{1}{z_1} + \frac{1}{z_2} + ... + \frac{1}{z_n} \right)$$

is real and $0 \leq \omega \leq n^2$.

8 (3 pts). Let $f : [0, 1] \to \mathbb{R}$ be a decreasing function. Prove that for any $a \in (0, 1)$

$$a \int_0^1 f(x)\, dx \leq \int_0^a f(x)\, dx.$$

9 (4 pts). Let $(x_n)_{n\geq 1}$ be a sequence of real numbers satisfying

$$x_{n+m} \leq x_n + x_m \qquad \text{for all } n, m \geq 1.$$

Prove that

$$\lim_{n\to\infty} \frac{x_n}{n} \quad \text{exists and is finite.}$$

10 (4 pts). Which of

$$\sqrt{2+\sqrt{3+\sqrt{2+...}}} \text{ or } \sqrt{3+\sqrt{2+\sqrt{3+...}}}$$

is larger? Each number contains n succesive square roots.