Math 3974 Problem Seminar Homework 2

Due February 20, 2017

There are hints at the end of this file. Also, for those problems which are Putnam problems, you can find solutions at `http://kskedlaya.org/putnam-archive/`. However, we suggest you at least spend 10 minutes on the problem before reading the solution, and you must write your own solution.

Problem 1.1. (Difficulty:1) Consider the polynomial

$$f(x) = (x-a_1)^{k_1} \cdots (x-a_n)^{k_n}.$$

Show that

$$\big(\log f(x)\big)' = \frac{k_1}{x-a_1} + \cdots + \frac{k_n}{x-a_n}.$$

Problem 1.2. (Difficulty:1) Let $p(x) = x^3 + ax^2 + bx + c$ be a polynomial with zeros α, β, and γ. Find a degree 4 monic polynomial whose zeros are $\alpha+\beta+1$, $\beta+\gamma+1$, and $\gamma+\alpha+1$.

Problem 1.3. (Difficulty: 2) Let $p(x) = a_n x^n + \cdots + a_0$ be a *real coefficient* polynomial which satisfies $p(\zeta x) = p(x)$ for $\zeta = e^{2\pi i/3} = \frac{-1\pm\sqrt{-3}}{2}$. Show that $a_i = 0$ if $3 \nmid i$.

Problem 1.4. (Difficulty:3) Using the heuristic argument of Vieta's formula for $\sin x$, show that

$$\sum_{n\geq 1} \frac{1}{n^4} = \frac{\pi^4}{90}.$$

Problem 1.5. (Difficulty:4) Let k be a fixed positive integer. The n-th derivative of $\frac{1}{x^k-1}$ has the form $\frac{P_n(x)}{(x^k-1)^{n+1}}$ where $P_n(x)$ is a polynomial. Find $P_n(1)$.

Problem 1.6. (Difficulty:4) [variant of 1991-A1] Find polynomials $f(x)$ and $g(x)$ such that

$$|f(x)| + g(x) = \begin{cases} 1 & \text{if } x < 0 \\ -2x+1 & \text{if } x > 0. \end{cases}$$

Problem 1.7. (1) (Difficulty:2) Let $p(x)$ be the polynomial $(x-a)(x-b)$ with two distinct integers a and b. Suppose that $p(x) = 1$ has an integral root k. Show that (k, a, b) are consecutive integers (not necessarily in this order).

(2) (Difficult:2 once you did (1)) Let $p(x)$ be the polynomial $(x-a)(x-b)(x-c)(x-d)$ with a, b, c, d distinct integers. Assume that $p(x) = 4$ has an integral root k. Show that k is the mean of a, b, c, d.

Problem 1.8. (Difficulty:4) If $P(x)$ is a polynomial of degree n such that $P(k) = k/(k+1)$ for $k = 0, \ldots, n$, determine $P(n+1)$.

Problem 1.9. (Difficulty:2) [2016-A1] Find the smallest positive integer j such that for every polynomial $p(x)$ with integer coefficients and for every integer k, the integer

$$p^{(j)}(k) = \frac{d^j}{dx^j} p(x)\big|_{x=k}$$

(the jth derivative of $p(x)$ at k) is divisible by 2016.

Hints:

Problem 1.3: If two polynomials are equal, then their real part and the complex part must be equal.

Problem 1.5: Use induction on n.

Problem 1.8: Instead of looking for $P(x)$, try to determine $(x+1)P(x) - x$.