Math 3974 Problem Seminar Homework 1

Due September 13, 2016

Problem 7.1. (Difficulty:1+2+1) Find the ordinary power series $\sum_{n\geq 0} a_n x^n$ generating functions of each of the following sequence, in simple, closed form. In each case the sequence is defined for all $n \geq 0$.

(a) $a_n = n^2$
(b) $a_n - P(n)$, where P is a given polynomial, of degree m
(c) $a_n = 5 \cdot u^n - 3 \cdot 4^n$.

Problem 7.2. (Difficulty:1+1+1+1) If $f(x) = \sum_{n\geq 0} a_n x^n$ is the ordinary power series generating function of the sequence $\{a_n\}_{n\geq 0}$, then express simply, in terms of $f(x)$, the ordinary power series generating functions of the following sequence (start with $n = 0, 1, 2...$)

(a) $\{na_n + c\}$
(b) $\{0, 0, 1, a_3, a_4, a_5, \dots\}$
(c) $a_0, 0, a_2, 0, a_4, 0, \dots$
(d) $\{a_{n+2} - a_{n+1} - a_n\}$

Problem 7.3. (Difficulty:1) Find the x^n-coefficient of $\dfrac{1}{(1-ax)(1-bx)}$ where $a \neq b$.

Problem 7.4. (Difficulty:2) Find the ordinary power series generating function of the sequence

$$a_0 = 0,\ a_1 = 1,\ a_{n+2} = 3a_{n+1} - 2a_n \text{ for } n \geq 0.$$

Problem 7.5. (Difficulty:3) Let $f(n)$ be the number of subsets of $\{1, \dots, n\}$ that contain no two consecutive elements, for positive integer n. Find the recurrence that is satisfied by these numbers, and then give a closed formula for these numbers $f(n)$.

Problem 7.6. (Difficulty:4) Let $x^{(n)} = x(x-1)\cdots(x-n+1)$ for n a positive integer, and let $x^{(0)} = 1$. Prove that

$$(x+y)^{(n)} = \sum_{k=0}^{n} \binom{n}{k} x^{(k)} y^{(n-k)}.$$

Problem 7.7. (Difficulty:3) A function f is defined for all $n \geq 1$ by the relation

(a) $f(1) = 1$
(b) $f(2n) = f(n)$
(c) $f(2n+1) = f(n) + f(n+1)$.

Let

$$F(x) = \sum_{n\geq 1} f(n) x^{n-1}$$

be the generating function of the sequence. Show that

$$F(x) = (1 + x + x^2) F(x^2),$$

and conclude that

$$F(x) = \prod_{j\geq 0} \left\{1 + x^{2^j} + x^{2^{j+1}}\right\}.$$

Problem 7.8. (Difficulty:7) Sum the series

$$\sum_{m\geq 1} \sum_{n\geq 1} \frac{3^{-m} m^2 n}{3^m n + 3^n m}.$$